Hw 1

1. Given an ordered set $(S,<)$, $\phi \neq E \subset S$, and $y \in S$ is an upper (lower) bound of E.

Prove that if $y \in E$, then $y = \sup(\inf) E$. The element y is called the maximal (minimal) element of E.

<u>Def</u>: S is an ordered set, $E \subset S$, and E is bounded above.

Suppose there exists an $\alpha \in S$ with the following properties:

(1) α is an upper bound of E.

(2) If $\gamma < \alpha$, then γ is not an upper bound of E.

Then α is called the supremum of E, $\alpha = \sup E$. <u>Rmk</u>: α is unique.

<pf>

① Since $y \in S$ is an upper bounded of E, then E is bounded above.

Claim that $y = \sup E$.

Clearly, (1) holds.

Let $x < y$ for any $x \in S$.

Claim that x is <u>not</u> an upper bound of E, that is, $\exists z \in E$ such that $x < z$.

Since $y \in E$ and $x < y$, then x is <u>not</u> an upper bound of E.

So (2) holds.

Thus, $y = \sup E$.

②

Since $y \in S$ is a lower bound of E, then E is bounded below.

Claim that $y = \inf E$, that is, (1) y is a lower bound of E

(2) If $y < x$, then x is not a lower bound of E.

Clearly, (1) holds. Let $y < x$ for some $x \in S$.

Claim that x is not a lower bound of E, that is, $\exists\, z \in E$ such that $z < x$.

Since $y \in E$ and $y < x$, then x is not a lower bound of E. ■

2. Prove, that for an ordered set $(S, <)$, if S has GLB (Greatest Lower Bound Property), then S has LUB (Least Upper bound Property). We therefore Conclude that LUB and GLB are two equivalent properties.

Def: ① An ordered set S has LUB, if $E \neq \phi$, $E \subset S$, E is bounded above, then $\sup E$ exists in S.

② An ordered set S has GLB, if $E \neq \phi$, $E \subset S$, E is bounded below, then $\inf E$ exists in S.

<pf> Let S has GLB, that is, if $E \neq \phi$, $E \subset S$, E is bounded below, then $\inf E$ exists in S.

Given $E \neq \phi$, $E \subset S$, E is bounded above.

Let L be the set of all upper bound of E.

Since E is bounded above, then there is a $x \in S$ such that x is an upper bound of E. Thus $L \neq \phi$.

For any $y \in L$, then y is an upper bound of E, that is, for any $y \in L$, we have $z \leq y$ for every $z \in E$.

Thus, for every $z \in E$ is a lower bound of L.

Since $E \neq \phi$, then we choose $z \in E$ such that z is a lower bound of L.

Thus, L is bounded below.

Since S has GLBP, then $\inf L$ exists in S.

Let $\alpha = \inf L$.

Claim that $\sup E = \alpha$, that is, (1) α is an upper bound of E. ($\alpha \in L$)

(2) $\forall \gamma < \alpha$, then γ is not an upper bound of E.
($\gamma \notin L$)

(1) If $\gamma > \alpha$, since $\alpha = \inf L$, then γ is not a lower bound of L.

Hence $\gamma \notin E$.

[Suppose $\gamma \in E$. Since $\alpha = \inf L$ and $\gamma > \alpha$, then $\exists z \in L$ s.t. $\alpha < z < \gamma$.

Since $z \in L$, that is, z is an upper bound of E, but $\gamma \in E$ and $z < \gamma$.

This is a contradiction. Thus, $\gamma \notin E$.]

Thus, we have that if $\gamma > \alpha$, then $\gamma \notin E \Leftrightarrow$ if $\gamma \in E$, then $\gamma \leq \alpha$.

Since γ is arbitary, so $\forall \gamma \in E$, we have $\gamma \leq \alpha$, that is, α is an upper bound of E.
($\alpha \in L$.)

(2) If $\gamma < \alpha$, then $\gamma \notin L$.

[Suppose $\gamma \in L$. Since $\alpha = \inf L$, then α is a lower bound of L.

But we have $\gamma < \alpha$, $\gamma \in L$, α is a lower bound of L. This is a contradiction.

Thus, $\gamma \notin L$.]

Now, we have that $\underline{\alpha \in L \text{ } (\because (1) \text{ holds})}$ and $\underline{\text{if } \gamma < \alpha, \text{ then } \gamma \notin L}$, that is,

$\underline{\alpha \text{ is an upper bound of } E}$ and $\underline{\text{if } \gamma < \alpha, \text{ then } \gamma \text{ is not an upper bound of } E.}$

Thus, $\alpha = \sup E$.

3. Rudin, Ex 1 (p21) and Ex 2. (p22)

Ex 1.
If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

Ex 2. Prove that there is no rational number whose square is 12.

<pf Ex 1>

① Suppose $r+x$ is rational.

Then $x = (r+x) - r$ must be rational. (This is an contradiction)

Thus $r+x$ is irrational.

② Suppose rx is rational.

Then $x = \frac{rx}{r}$, $r \neq 0$ must be rational. (This is an contradiction)

Thus rx is irrational.

<pf of Ex 2>

$\sqrt{12} = 2 \cdot \sqrt{3}$

Since 2 is rational and we claim that $\sqrt{3}$ is irrational, by Ex 1,

then $\sqrt{12} = 2 \cdot \sqrt{3}$ is rational.

Suppose $\sqrt{3}$ is rational. Let $\sqrt{3} = \frac{m}{n}$, $n \neq 0$, $(m,n) = 1$, $m, n \in \mathbb{Z}$.

Then $m^2 = 3n^2$. So $3 \mid m^2$.

Let $m = 3k+1$ for some $k \in \mathbb{Z}$. Then $m^2 = 9k^2 + 6k + 1$, so $3 \nmid m^2$. (contradiction)

Let $m = 3k+2$ for some $k \in \mathbb{Z}$. Then $m^2 = 9k^2 + 12k + 4$, so $3 \nmid m^2$. (contradiction)

Thus $m = 3k$ for some $k \in \mathbb{Z}$ (that is, $3 \mid m$)

Then $(3k)^2 = 3 \cdot n^2$. So $3 \mid n^2$.

By above similar method, then $n = 3l$ for some $l \in \mathbb{Z}$.

But $(m, n) = 1$, this is an contradiction.

Thus, $\sqrt{3}$ is irrational. Then $\sqrt{12} = 2 \cdot \sqrt{3}$ is irrational. $\blacksquare$

4. Rudin, Ex5 (p22)

Ex 5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$.

<pf> Let $-A = \{x \mid x \in A\}$. Since $A \neq \phi$, then $-A \neq \phi$.

Since A is bounded below, then there is a $y \in \mathbb{R}$ such that $y \leq x$, $\forall x \in A$.

Then $-x \leq -y$, $\forall -x \in (-A)$, that is, $-A$ is bounded above.

Since $\mathbb{R}$ is an ordered set, $A \subset \mathbb{R}$, $-A \subseteq \mathbb{R}$, A is bounded below, $-A$ is bounded above, then $\inf A$ and $\sup(-A)$ exists.

Let $\alpha = \inf A$ and $\beta = \sup(-A)$.

claim: $\inf A = -\beta$, that is, $-\beta$ is the greatest lower bound of A.

① Since $x \in A$, then $-x \in (-A)$. Since $\beta = \sup(-A)$, then $-x \leq \beta$, $\forall -x \in (-A)$.

So $-\beta \leq x$, $\forall x \in A$, that is, $-\beta$ is a lower bound of A.

② Let $-\beta < \gamma$. Claim that γ is not a lower bound of A.

Suppose γ is a lower bound of A. Then $\gamma \leq x$, $\forall x \in A$.

Since $-x \leq -\gamma$, $\forall -x \in (-A)$, then $-\gamma$ is a upper bound of $-A$.

Since $\beta = \sup(-A)$, then $\beta \leq -\gamma$. So $\gamma \leq -\beta$. (This is a contradiction)

Thus, γ is not a lower bound of A.

By definition of $\inf A$, then $\inf A = -\beta$, that is, $\inf A = -\sup(-A)$.

5. Prove, using $\varepsilon-\delta$ definition, that

(a) $$\lim_{x\to 1}\frac{x}{x+1}=\frac{1}{2}$$

(b) $$\lim_{x\to\infty}\frac{2x^2}{4x^2+3x-1}=\frac{1}{2}.$$

<pf>

(a) $$\left|\frac{x}{x+1}-\frac{1}{2}\right|=\left|\frac{x-1}{2(x+1)}\right|=\frac{|x-1|}{|2\cdot(x+1)|}$$

Choose $|x-1|<1$, then $0<x<2$.

Thus we have $2<2(x+1)<6 \Rightarrow \frac{1}{6}<\frac{1}{2(x+1)}<\frac{1}{2} \Rightarrow \frac{1}{|2(x+1)|}<\frac{1}{2}$.

Given $\varepsilon>0$, $\exists\,\delta=\min\{1,2\varepsilon\}>0$, such that if $0<|x-1|<\delta$, then

$$\left|\frac{x}{x+1}-\frac{1}{2}\right|=\frac{|x-1|}{|2(x+1)|}<\frac{1}{2}|x-1|<\frac{1}{2}\cdot\delta<\frac{1}{2}\cdot 2\varepsilon=\varepsilon. \quad \blacksquare$$

(b) $$\left|\frac{2x^2}{4x^2+3x-1}-\frac{1}{2}\right|=\left|\frac{-3x+1}{2(4x^2+3x-1)}\right|=\frac{|-3x+1|}{|2(4x^2+3x-1)|}$$

① Choose $x>1$, then $-3x+1<-2<0$, $2(4x^2+3x-1)>12>0$.

Thus we have $$\frac{|-3x+1|}{|2(4x^2+3x-1)|}=\frac{3x-1}{8x^2+6x-2}<\frac{3x}{8x^2}=\frac{3}{8x}<\frac{1}{2x}.$$

Since $3x-(3x-1)=1>0$, then $3x>3x-1>2>0$.

Since $(8x^2+6x-2)-8x^2=6x-2>4>0$, then $8x^2+6x-2>8x^2>8>0$.

Given $\varepsilon>0$, $\exists\,M=\max\{1,\frac{1}{2\varepsilon}\}>0$, such that if $x>M$, then

$$\left|\frac{2x^2}{4x^2+3x-1}-\frac{1}{2}\right|=\frac{|-3x+1|}{|8x^2+6x-2|}<\frac{1}{2x}<\frac{1}{2M}<\frac{1}{2}\cdot 2\varepsilon=\varepsilon.$$

(2) Choose $x < -2$, then $-3x+1 > 7 > 0$ and $2(4x^2+3x-1) > 9 > 0$.

Then we have $\dfrac{|-3x+1|}{|2(4x^2+3x-1)|} = \dfrac{-3x+1}{8x^2+6x-2} < \dfrac{-4x}{7x^2} = \dfrac{-4}{7x}$.

since $(8x^2+6x-2) - 7x^2 = x^2+6x-2 = (x+3)^2 + 7 > 7 > 0$, then $8x^2+6x-2 > 7x^2$.

since $(-4x) - (-3x+1) = -x-1 > 1 > 0$, then $-3x+1 < -4x$.

Given $\varepsilon > 0$, $\exists\ N = \min\left\{-2, \dfrac{-4}{7\varepsilon}\right\} < 0$, such that if $x < N$, then

$$\left| \frac{2x^2}{4x^2+3x-1} - \frac{1}{2} \right| = \frac{|-3x+1|}{|8x^2+6x-2|} < \frac{-4}{7x} < \varepsilon.$$

$\left(\text{since } x < N < \dfrac{-4}{7\varepsilon} \Rightarrow 7\varepsilon x < -4,\ x < 0 \Rightarrow \varepsilon > \dfrac{-4}{7x} \right)$

■

6. Prove, using $\varepsilon-\delta$ definition, that if $f(x) \to L$ and $g(x) \to M \neq 0$ as $x \to c$,

then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

<pf>

$$\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| = \frac{|M \cdot f(x) - L \cdot g(x)|}{|g(x) \cdot M|} = \frac{|M \cdot f(x) - f(x) \cdot g(x) + f(x) \cdot g(x) - L \cdot g(x)|}{|g(x)| \cdot |M|}$$

$$\leq \frac{|f(x)| \cdot |g(x) - M|}{|g(x)| \cdot |M|} + \frac{|f(x) - L| \cdot |g(x)|}{|M| \cdot |g(x)|}$$

Given $\varepsilon > 0$, since $\lim_{x \to c} f(x) = L$, then there is $\delta_1 > 0$ such that if $0 < |x-c| < \delta_1$,

we have $|f(x) - L| < \frac{|M|}{2} \cdot \varepsilon$, $M \neq 0$.

Given $\varepsilon = 1 > 0$, since $\lim_{x \to c} f(x) = L$, then there is $\delta_2 > 0$ such that if $0 < |x-c| < \delta_2$,

we have $|f(x) - L| < 1. \Rightarrow -1 + L < f(x) < 1 + L$.

Let $N_1 = \max\{|1+L|, |-1+L|\} > 0$. Then $|f(x)| \leq N_1$, if $0 < |x-c| < \delta_2$.

Given $\varepsilon = \frac{|M|}{2} > 0$, since $\lim_{x \to c} g(x) = M$, then there is $\delta_3 > 0$ such that if $0 < |x-c| < \delta_3$,

we have $|g(x) - M| < \frac{|M|}{2}$.

$$\Rightarrow -\frac{|M|}{2} < g(x) - M < \frac{|M|}{2}$$

$$\Rightarrow M - \frac{|M|}{2} < g(x) < \frac{|M|}{2} + M.$$

Let $N_2 = \min\left\{\left|M - \frac{|M|}{2}\right|, \left|M + \frac{|M|}{2}\right|\right\} > 0$. Then $|g(x)| \geq N_2$, if $|x-c| < \delta_3$.
$(\because M \neq 0)$

$$\Rightarrow \frac{1}{|g(x)|} \leq \frac{1}{N_2}, \text{ if } |x-c| < \delta_3.$$

For this $\varepsilon > 0$, since $\lim_{x \to c} g(x) = M$, then there is $\delta_4 > 0$ such that if $0 < |x-c| < \delta_4$,

we have $$|g(x) - M| < \frac{N_2 \cdot |M|}{N_1} \cdot \frac{\varepsilon}{2}.$$

Now, given $\varepsilon > 0$, $\exists \delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4\} > 0$, such that if $0 < |x-c| < \delta$,

then
$$\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| = \frac{|M \cdot f(x) - L \cdot g(x)|}{|g(x)| \cdot |M|} = \frac{|f(x) \cdot (M - g(x)) + g(x) \cdot (f(x) - L)|}{|g(x)| \cdot |M|}$$

$$\leq \frac{|f(x)| \cdot |M - g(x)|}{|g(x)| \cdot |M|} + \frac{|g(x)| \cdot |f(x) - L|}{|g(x)| \cdot |M|}$$

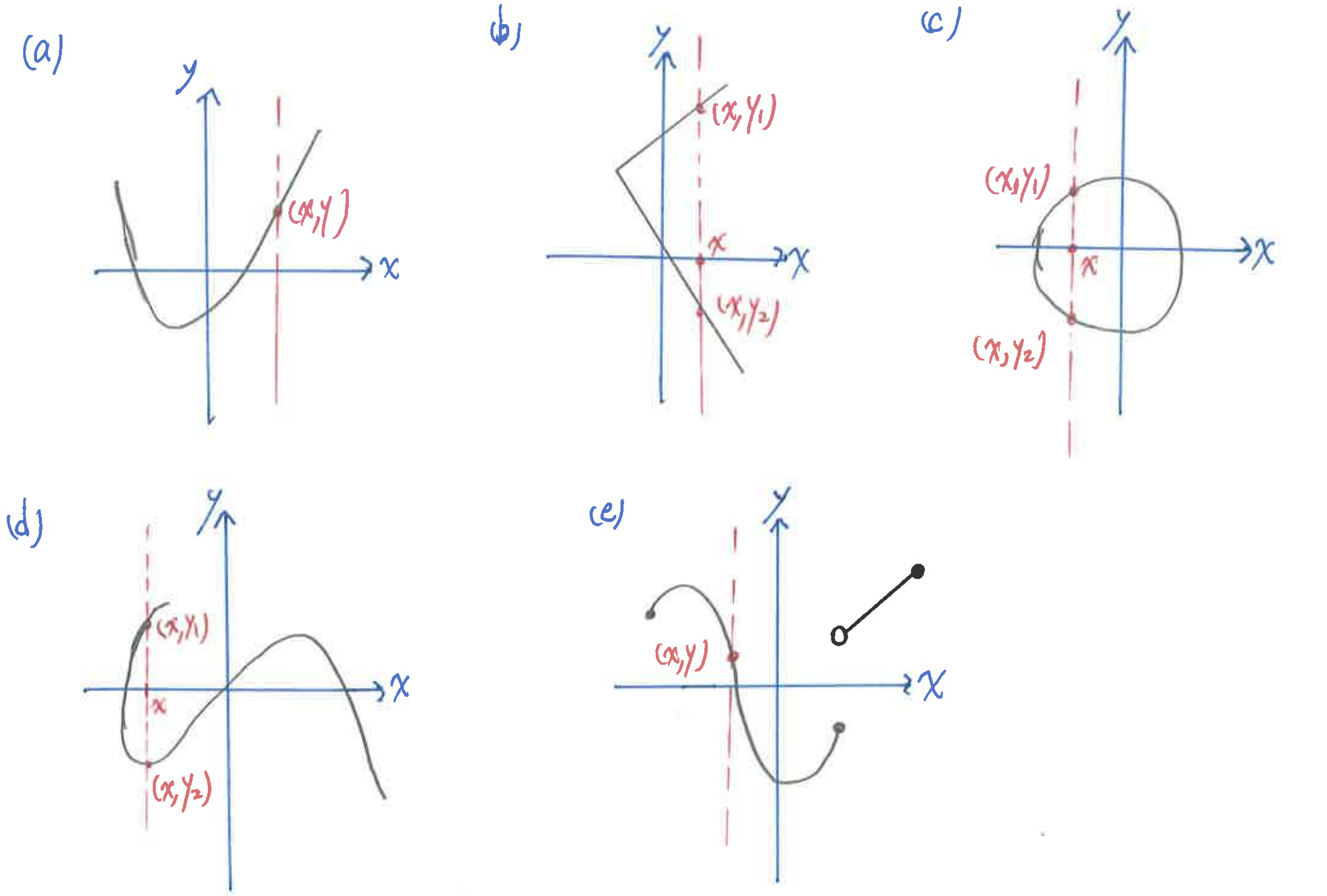
$$< \frac{N_1}{|M|} \cdot \frac{1}{N_2} \cdot \frac{N_2 \cdot |M|}{N_1} \cdot \frac{\varepsilon}{2} + \frac{1}{|M|} \cdot \frac{|M|}{2} \cdot \varepsilon$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

Thus, $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$. $\blacksquare$

7. Determine with sufficient reasons, whether each of the following curves is a graph of some function.



By the definition of $f(x)$: Let $y = f(x)$. Given a x, then we obtain only one $y = f(x)$.

By above check, we know that a graph of some function: $\begin{cases} \text{Yes} : (a), (e) \\ \text{No} : (b), (c), (d) \end{cases}$

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8. Salas §2-3: 18, 21, 26, 30, 43, 44, 47, 49, 50.

(建立在同學們已經輕鬆"極限的四則運算"的計算)

18.
$$\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}$$

21.
$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}=\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{4}$$

26.
$$\lim_{h\to 0}\frac{1-\frac{1}{h^2}}{1+\frac{1}{h^2}}=\lim_{h\to 0}\frac{h^2-1}{h^2+1}=\frac{-1}{1}=-1$$

30.
$$\lim_{\substack{x\to 2^+\\(x>2)}}\frac{\sqrt{x^2-4}}{x-2}=\lim_{x\to 2^+}\sqrt{\frac{(x+2)(x-2)}{(x-2)^2}}=\lim_{x\to 2^+}\sqrt{\frac{x+2}{x-2}}=\lim_{x\to 2^+}\frac{\sqrt{x+2}}{\sqrt{x-2}}=$$

Since $\lim_{x\to 2^+}\sqrt{2+x}=2$, $\lim_{x\to 2^+}\sqrt{x-2}=0$, so the limit does not exist.

43.

Let $f(x)=\begin{cases}1, & x\geq 0\\ 0, & x<0\end{cases}$ $\quad g(x)=\begin{cases}-1, & x\geq 0\\ 0, & x<0.\end{cases}$

Then $f(x)+g(x)=\begin{cases}0, & x\geq 0\\ 0, & x<0.\end{cases}$

Thus, we have $\lim_{x\to 0}f(x)$ does not exist, $\lim_{x\to 0}g(x)$ does not exist, but

$$\lim_{x\to 0}(f(x)+g(x))=0.$$

44. Let $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$, $g(x) = \begin{cases} -1, & x \geq 0 \\ 1, & x < 0. \end{cases}$

Then $f(x) \cdot g(x) = \begin{cases} -1, & x \geq 0 \\ -1, & x < 0. \end{cases}$

Thus, we have $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ does not exist, but $\lim_{x \to 0} [f(x) \cdot g(x)] = -1$.

47. If $\lim_{x \to c} \sqrt{f(x)}$ exists, then $\lim_{x \to c} f(x)$ exists.

<pf>

$\lim_{x \to c} f(x) = \lim_{x \to c} \left[\sqrt{f(x)} \cdot \sqrt{f(x)} \right] = \lim_{x \to c} \sqrt{f(x)} \cdot \lim_{x \to c} \sqrt{f(x)}$ exists. $\blacksquare$

49. If $\lim_{x \to c} f(x)$ exists, then $\lim_{x \to c} \frac{1}{f(x)}$ exists.

<sol>

Since $\lim_{x \to c} 1 = 1$ and $\lim_{x \to c} f(x)$ exists and "is not zero", then $\lim_{x \to c} \frac{1}{f(x)}$ exists.

So, this statement is false.

50. If $f(x) \leq g(x)$ for all $x \neq c$, then $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$.

<sol>

Let $g(x) = \begin{cases} 1, & x \neq 0 \\ -2, & x = 0 \end{cases}$ and $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0. \end{cases}$

Let $c = 0$. Then $f(x) \leq g(x)$ for all $x \neq c$. But $\lim_{x \to 0} g(x) = 1$ and $\lim_{x \to 0} f(x)$ does not exist.

9. Salas §2-5 : 5, 8, 13, 28, ~~35, 38, 39, 50.~~

5.

$$\lim_{x\to 0}\frac{\sin 4x}{\sin 2x}=\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{1}{\frac{\sin 2x}{2x}}\cdot 2 = 1\times\frac{1}{1}\times 2=\underline{2.}$$

Since $\lim\limits_{x\to 0}\frac{\sin 4x}{4x}=1$, $\lim\limits_{x\to 0}\frac{\sin 2x}{2x}=1$, $\lim\limits_{x\to 0} 2=2$.

8.

$$\lim_{x\to 0}\frac{\sin x^2}{x^2}=\lim_{t\to 0}\frac{\sin t}{t}=\underline{1}$$

Let $t=x^2$. Since $x\to 0$, then $t=x^2\to 0$.

13.

$$\lim_{x\to 0}\frac{2x}{\tan 3x}=\lim_{x\to 0}\frac{1}{\frac{\sin 3x}{3x}}\cdot\cos 3x\cdot\frac{2x}{3x}=\frac{1}{1}\times 1\times\frac{2}{3}=\underline{\frac{2}{3}.}$$

Since $\lim\limits_{x\to 0}\frac{\sin 3x}{3x}=1$, $\lim\limits_{x\to 0}\cos 3x=1$, $\lim\limits_{x\to 0}\frac{2x}{3x}=\lim\limits_{x\to 0}\frac{2}{3}=\frac{2}{3}$.

28.

$$\lim_{x\to 0}\frac{\sin^2 x}{x\cdot(1-\cos x)}=\lim_{x\to 0}\frac{\frac{\sin x}{x}\cdot\frac{\sin x}{x}}{\frac{1-\cos x}{x}}$$ does not exist.

Since $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$, $\lim\limits_{x\to 0}\frac{1-\cos x}{x}=0$.