Hw 2

1. Rudin Ch3, EX 1

Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$.

Is the converse true?

<pf> Let $\{s_n\} \subset \mathbb{R}$. (若有學過柯西數列的同学,不然背面有另解)

Since $\{s_n\}$ is convergent sequence, then $\{s_n\}$ is a Cauchy squence.

Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $|s_n - s_m| < \varepsilon$, as $n, m \geq N$.

For this $\varepsilon > 0$, $\exists N \in \mathbb{N}$, then we have

$$\big|\,|s_n| - |s_m|\,\big| \leq |s_n - s_m| < \varepsilon, \text{ as } n, m \geq N.$$

Thus, $\{|s_n|\}$ is also a Cauchy sequence.

By the completeness of the real number, then $\{|s_n|\}$ is a convergent sequence. $\blacksquare$

<sol>

Let $s_n = (-1)^n$ for all $n \in \mathbb{N}$.

Then $|s_n| = 1$ for all $n \in \mathbb{N}$. Thus $\{|s_n|\}$ is convergent sequence, but $\{s_n\}$ is not convergent sequence.

#

<pf>

Let $\lim_{n\to\infty} S_n = \alpha$, $\alpha \in \mathbb{R}$.

Given $\varepsilon > 0$, since $\lim_{n\to\infty} S_n = \alpha$, then there is a $N \in \mathbb{N}$ such that

$$|S_n - \alpha| < \varepsilon, \text{ as } n \geq N.$$

Then we have $\big||S_n| - |\alpha|\big| \leq |S_n - \alpha| < \varepsilon$, as $n \geq N$.

Thus, $\lim_{n\to\infty} |S_n| = |\alpha|$, that is, $\{|S_n|\}$ is also convergent sequence. $\blacksquare$

2. Rudin Ch3, EX 2

Calculate $\lim_{n\to\infty} (\sqrt{n^2+n} - n)$.

<sol>

$$\sqrt{n^2+n} - n = \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}$$

$$\lim_{n\to\infty} (\sqrt{n^2+n} - n) = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, then $\lim_{n\to\infty} (1+\frac{1}{n}) = 1$. Thus $\lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = 1$.

So $\lim_{n\to\infty} (\sqrt{n^2+n} - n) = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{1+1} = \frac{1}{2}$

※

3. Rudin Ch3, EX3.

If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2+\sqrt{s_n}}$, $n=1,2,3,\cdots\cdots$,

prove that $\{s_n\}$ converges, and that $s_n < 2$ for all $n=1,2,3,\cdots$.

<pf>

① Claim that $s_n < 2$ for all $n=1,2,3,\cdots$.

$n=1$, $s_1 = \sqrt{2} < 2$.

Suppose $n=k$ holds, that is, $s_k < 2. \Rightarrow \sqrt{s_k} < \sqrt{2} < 2$.

As $n=k+1$, then we have $s_{k+1} = \sqrt{2+\sqrt{s_k}} < \sqrt{2+2} = 2$.

By induction on n, then $s_n < 2$ for all $n=1,2,3,\cdots$, that is, s_n is bounded above by 2.

② Claim that s_n is increasing sequence, that is, $s_{n+1} \geq s_n$ for all $n=1,2,3,\cdots$

As $n=1$, then $s_2 = \sqrt{2+\sqrt{s_1}} > \sqrt{2} = s_1$.

Suppose $n=k$ holds, that is, $s_{k+1} \geqslant s_k$.

Let $n=k+1$. Then $s_{k+2} = \sqrt{2+\sqrt{s_{k+1}}} \geq \sqrt{2+\sqrt{s_k}} = s_{k+1}$.

By induction on n, then we have $s_{n+1} \geq s_n$ for all $n=1,2,3,\cdots$.

By ①,②, since $\{s_n\}$ is a increasing sequence, bounded above by 2,

then $\{s_n\}$ converges. ∎

4、 Salas §2-5 ※ 35、38、39、50.

35、

$$\lim_{x\to 0}\frac{1-\cos ax}{bx} = \lim_{x\to 0}\left(\frac{1-\cos ax}{ax}\cdot\frac{a}{b}\right) = 0\times\frac{a}{b} = \underline{0}, \quad \text{where } a\neq 0,\ b\neq 0.$$

Since $\lim\limits_{x\to 0}\dfrac{1-\cos ax}{ax} = 0$, $\lim\limits_{x\to 0}\dfrac{a}{b} = \dfrac{a}{b}$, $b\neq 0$.

※

38、 Show that if $\lim\limits_{x\to 0} f(x) = L$, then $\lim\limits_{x\to 0} f(ax) = L$ for each $a\neq 0$.

<pf>

Given $\varepsilon > 0$, since $\lim\limits_{x\to 0} f(x) = L$, then there is a $\delta_1 > 0$ such that

if $0 < |x-0| < \delta_1$, we have $|f(x) - L| < \varepsilon$.

Choose $\delta = \dfrac{\delta_1}{|a|} > 0$, if $0 < |x-0| < \delta = \dfrac{\delta_1}{|a|}$, then $0 < |ax-0| < \delta_1$.

Since $0 < |ax-0| < \delta_1$, then we have $|f(ax) - L| < \varepsilon$.

$\blacksquare$

39、 $f(x) = \sin x$, $c = \frac{\pi}{4}$

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{\sin(\frac{\pi}{4}+h) - \sin\frac{\pi}{4}}{h}$$

$$= \lim_{h\to 0} \frac{\sin\frac{\pi}{4}\cosh + \cos\frac{\pi}{4}\sinh - \sin\frac{\pi}{4}}{h}$$

$$= \lim_{h\to 0} \left[\sin\frac{\pi}{4}\cdot\left(\frac{\cosh-1}{h}\right) + \cos\frac{\pi}{4}\cdot\frac{\sinh}{h}\right]$$

Since $\lim_{h\to 0} \frac{\cosh-1}{h} = \lim_{h\to 0}\left[(-1)\cdot\frac{1-\cosh}{h}\right] = -1\times 0 = 0$ and

$\lim_{h\to 0}\frac{\sinh}{h} = 1$, then $\lim_{h\to 0} \frac{\sin(\frac{\pi}{4}+h) - \sin\frac{\pi}{4}}{h} = \sin\frac{\pi}{4}\times 0 + \cos\frac{\pi}{4}\times 1 = \underline{\frac{\sqrt{2}}{2}}$ ※

50、 Given that $\lim_{x\to c} f(x) = 0$ and $|g(x)| \le B$ for all x in an interval $(c-p, c+p)$,

prove that $\lim_{x\to c} f(x)g(x) = 0$.

<pf> Case 1: $B \neq 0$.

Given $\varepsilon > 0$, since $\lim_{x\to c} f(x) = 0$, then there is a $\delta_1 > 0$ such that

if $0 < |x-c| < \delta_1$, we have $|f(x) - 0| < \varepsilon/B$.

Choose $\delta = \min\{\delta_1, p\} > 0$.

If $0<|x-c|<\delta$, then $|f(x)g(x) - 0| = |f(x)|\cdot|g(x)| \le B\cdot|f(x)| < B\cdot\frac{\varepsilon}{B} = \varepsilon$.

case 2: $B = 0$. Then $g(x) = 0$ for all $c-p < x < c+p$.

Given $\varepsilon > 0$, $\exists\, \delta = p > 0$, such that if $0 < |x-c| < \delta$ $(c-p < x < c+p,\ x \neq c)$,

then $|f(x)g(x) - 0| = |f(x)|\cdot|g(x)| = 0 < \varepsilon$. ■

5. Salas §11-2 #40, 63.

40. $a_n = \dfrac{1-(\frac{1}{2})^n}{(\frac{1}{2})^n}$

① $\dfrac{a_{n+1}}{a_n} = \dfrac{1-(\frac{1}{2})^{n+1}}{(\frac{1}{2})^{n+1}} \cdot \dfrac{(\frac{1}{2})^n}{1-(\frac{1}{2})^n} = 2 \cdot \dfrac{1-(\frac{1}{2})^{n+1}}{1-(\frac{1}{2})^n}$

Since $(\frac{1}{2})^{n+1} < (\frac{1}{2})^n \Rightarrow 1-(\frac{1}{2})^{n+1} > 1-(\frac{1}{2})^n > 0 \Rightarrow 1 < \dfrac{1-(\frac{1}{2})^{n+1}}{1-(\frac{1}{2})^n}$,

then $\dfrac{a_{n+1}}{a_n} = 2 \cdot \dfrac{1-(\frac{1}{2})^{n+1}}{1-(\frac{1}{2})^n} > 2 > 1$, that is, $a_{n+1} > a_n$.

So, a_n is increasing.

② $a_1 = 1$, $a_2 = \dfrac{\frac{3}{4}}{\frac{1}{4}} = 3$, $a_3 = \dfrac{\frac{7}{8}}{\frac{1}{8}} = 7$, ..., clear, it is unbounded.

Suppose a_n is bounded above, i.e., $a_n \leq M$ for all n, for some $M>0$.

Then $a_n = \dfrac{1-(\frac{1}{2})^n}{(\frac{1}{2})^n} = 2^n - 1 \leq M$ for all n, for some $M>0$.

$\Rightarrow 2^n \leq M+1$ for all n

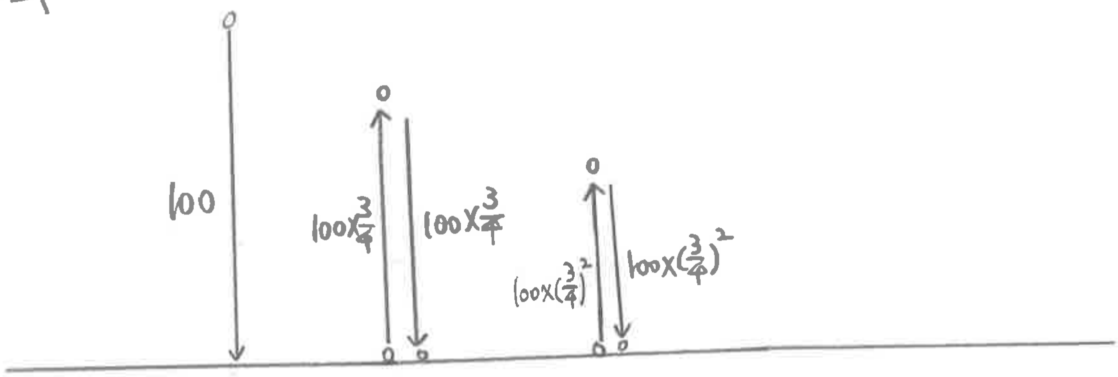
Choose $N = [\log_2 (M+1)] + 1 \in \mathbb{N}$. As $n \geq N$, then we have

$2^n \geq 2^N = 2^{[\log_2 (M+1)]+1} > 2^{\log_2 (M+1)} = M+1$. (This is a contradiction)

Thus, a_n is unbounded above.

Since a_n is increasing, then a_n is bounded below by $a_1 = 1$. #

63.



①

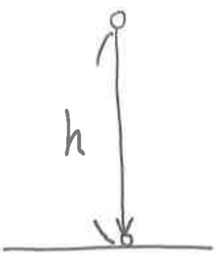
$$S_1 = 100 \times \frac{3}{4} \times 2 = 150$$

$$S_2 = 100 \times \left(\frac{3}{4}\right)^2 \times 2 = 150 \times \frac{3}{4}$$

$$\vdots$$

$$S_n = 100 \times \left(\frac{3}{4}\right)^n \times 2 = \underline{150 \times \left(\frac{3}{4}\right)^{n-1}}$$

②



$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2h}{32}} = \frac{1}{4}\sqrt{h} \quad , \quad g = 9.8\,m/s^2 = 32\ feet/s^2$$

$$t_n = 2 \times \frac{1}{4} \times \sqrt{\frac{S_n}{2}} = \frac{1}{2} \times \sqrt{75 \times \left(\frac{3}{4}\right)^{n-1}} = \underline{\frac{5\sqrt{3}}{2} \times \left(\frac{3}{4}\right)^{\frac{n-1}{2}}}$$

6. Salas §11-3 ※ 21, 28, 54, 59.

21.

$\lim_{n\to\infty} \cos(n\pi)$ does not converge.

Since n is odd, then $\cos(n\pi) = -1$.

Since n is even, then $\cos(n\pi) = 1$. ※

59. Prove that $\left(\frac{1}{n}\right)^{\frac{1}{p}} \to 0$ for all positive integers p.

$\lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{p}} = 0$, for all $p \in \mathbb{N}$.

<pf>

$$\left|\left(\frac{1}{n}\right)^{\frac{1}{p}} - 0\right| = \left(\frac{1}{n}\right)^{\frac{1}{p}} < \varepsilon \iff \frac{1}{n} < \varepsilon^p \iff n > \frac{1}{\varepsilon^p}$$

Given $\varepsilon > 0$, $\exists\ N = \left[\frac{1}{\varepsilon^p}\right] + 1 \in \mathbb{N}$, such that we have

$$\left|\left(\frac{1}{n}\right)^{\frac{1}{p}} - 0\right| = \left(\frac{1}{n}\right)^{\frac{1}{p}} \leq \left(\frac{1}{N}\right)^{\frac{1}{p}} < \left(\frac{1}{\frac{1}{\varepsilon^p}}\right)^{\frac{1}{p}} = \varepsilon,$$

if $n \geq N > \frac{1}{\varepsilon^p}$.

■

28.

$\lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ converges.

Since $\lim_{n\to\infty}\frac{1}{n}=0$, $\lim_{n\to\infty}\frac{1}{n+1}=0$, then $\lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=0-0=0$ ※

54. If $a_n\to L$, then $|a_n|\to|L|$. Is the converse true?

<sol> Answer: No

$$a_n=\begin{cases}1+\frac{1}{n}, & n=2,4,6,8,10,\cdots\cdots\\ -1-\frac{1}{n}, & n=1,3,5,7,9,\cdots\cdots\end{cases}$$

Then $|a_n|=1+\frac{1}{n}$, $n=1,2,3,4,5,\cdots\cdots$ and $\lim_{n\to\infty}|a_n|=1$.

But $\lim_{n\to\infty}a_n$ does not converge. ※