Names and Student IDs: ____________________

Homework 3 Calculus 1

1. Rudin Chapter 3, Exercise 3.
2. Rudin Chapter 3, Exercise 5. For simplification, assume that sequences $\{a_n\}$ and $\{b_n\}$ are both bounded.
3. Given a sequence $\{a_n\}$ of real numbers and define

$$b_k = \sup_{n \geq k} a_n.$$

Prove that
(a)

$$\lim_{k \to \infty} b_k = \inf_k b_k.$$

(b)

$$\lim_{k \to \infty} b_k = \limsup_n a_n.$$

Similar arguments show that (you don't have to do it)

$$\liminf_n a_n = \lim_{k \to \infty} \inf_{n \geq k} a_n = \sup_{k \to \infty} \inf_{n \geq k} a_n.$$

4. Given a nonempty subset $E \subset \mathbb{R}$ that is bounded above (so that $\sup E$ exists), prove that there exists a sequence $\{a_n\} \subset E$ converging to $\sup E$.
5. Given two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers with $a_n \leq b_n$. Prove that
   (a) $\limsup_n a_n \leq \limsup_n b_n$.
   (b) $\liminf_n a_n \leq \liminf_n b_n$.
   (c) If both sequences are convergent, then $\lim_n a_n \leq \lim_n b_n$.

   Also, give counterexample to the statement $\limsup_n a_n \leq \liminf_n b_n$.
6. If $a_n > 0$, and $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L > 0$, then $\lim_{n \to \infty} \sqrt[n]{a_n} = L$. (Note that the statement is also true for $L = 0$).

7. Find the limit of
   (a) $\sqrt[n]{n}$.
   (b) $\sqrt[n]{n^5 + n^4}$.
   (c) $\sqrt[n]{\frac{n!}{n^n}}$.

   Hint: Use problem 6.

8. Rudin Chapter 3, Exercise 4.

9. Salas 11.4: 7, 20, 23, 24.

10. Salas 12.2: 3, 7, 9, 14, 29.