Hw 4.

1. Prove that if a_k is a monotonic sequence and $\sum_{k=1}^{\infty} a_k$ converges,

then $\lim_{k\to\infty} k a_k = 0$.

<pf> Let $b_n = \sum_{k=1}^{n} a_k$. Since $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} b_n = \sum_{k=1}^{\infty} a_k$ converges.

So $\{b_n\}$ is a Cauchy sequence.

Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $|b_n - b_m| < \varepsilon$, if $n, m \geq N$.

Let $n \geq m$, then $|b_n - b_m| = \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon$, if $m, n \geq N$. —①

Let $\{a_k\}$ be an increasing sequence, that is, $a_n \leq a_m$, if $n \leq m$.

Since $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Since $\{a_k\}$ is an increasing sequence and $\lim_{k\to\infty} a_k = 0$, then $a_k < 0$ for all $k \in \mathbb{N}$.

So $k a_k < 0$ for all $k \in \mathbb{N}$.

By ①, if $k \geq N$, then $|b_k - b_N| = |a_{N+1} + a_{N+2} + \cdots + a_k| < \varepsilon$.

$\Rightarrow -\varepsilon < a_{N+1} + a_{N+2} + \cdots + a_k$, if $k \geq N$

$\Rightarrow -\varepsilon < a_{N+1} + a_{N+2} + \cdots + a_k \leq (k-N) \cdot a_k < k a_k < 0 < \varepsilon$, if $k \geq N$

$\Rightarrow |k a_k - 0| = |k a_k| < \varepsilon$, if $k \geq N$.

So, So, $\lim_{k\to\infty} k a_k = 0$, if $\{a_k\}$ is an increasing sequence.

Similarly, if $\{a_k\}$ is a decreasing sequence, then $\lim_{k\to\infty} k a_k = 0$.

∎

2. Rudin Chapter 3, #6 (a) (b) (c)

Investigate the behavior (convergence or divergence) of Σa_n,

if (a) $a_n = \sqrt{n+1} - \sqrt{n}$

(b) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$

(c) $a_n = (\sqrt[n]{n} - 1)^n$

<pf of (a)>

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \frac{1}{2\sqrt{n+1}} \geq \frac{1}{2(n+1)}, \text{ let } b_n = \frac{1}{2(n+1)}.$$

Since $\Sigma b_n = \Sigma \frac{1}{2(n+1)}$ diverges, $0 \leq b_n \leq a_n$ for all $n \in \mathbb{N}$,

by comparison test, then Σa_n diverges.

$\blacksquare$

<pf of (b)>

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{2n\sqrt{n}} = \frac{1}{2n^{3/2}}, \text{ let } b_n = \frac{1}{2n^{3/2}},$$

by p-series, then $\Sigma b_n = \Sigma \frac{1}{2n^{3/2}}$ converges.

Since Σb_n converges, $0 \leq a_n \leq b_n$ for all n, by comparison test,

then Σa_n converges.

$\blacksquare$

<pf of (c)>

Now, we know that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Given $\varepsilon = \frac{1}{2} > 0$, $\exists\, N \in \mathbb{N}$ such that if $n \geq N$, then $|\sqrt[n]{n} - 1| < \frac{1}{2}$

Since $\sqrt[n]{n} \geq 1$ for all $n \in \mathbb{N}$, then $0 \leq \sqrt[n]{n} - 1 < \frac{1}{2}$, if $n \geq N$.

Let $a_n = (\sqrt[n]{n} - 1)^n$. Then $a_n = (\sqrt[n]{n} - 1)^n < (\frac{1}{2})^n$, if $n \geq N$.

Since $\sum (\frac{1}{2})^n = \dfrac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 < \infty$, $0 \leq a_n < (\frac{1}{2})^n$ for all $n \in \mathbb{N}$,

by comparison test, then $\sum a_n$ converges.

$\blacksquare$

3. Rudin chapter 3, ※11 (a).

Suppose $a_n > 0$, $S_n = a_1 + a_2 + \cdots + a_n$ and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

<pf of (a)>

Suppose $\sum \frac{a_n}{1+a_n}$ converges. Then $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0$.

Given $\varepsilon = \frac{1}{2}$, $\exists N \in \mathbb{N}$, such that if $n \geq N$, then $\left| \frac{a_n}{1+a_n} - 0 \right| < \frac{1}{2}$

$$\Rightarrow 0 < \frac{a_n}{1+a_n} < \frac{1}{2}, \text{ if } n \geq N$$

$$\Rightarrow 0 < a_n < \frac{1}{2} + \frac{1}{2}a_n, \text{ if } n \geq N$$

$$\Rightarrow \frac{1}{2}a_n < \frac{1}{2}, \text{ if } n \geq N$$

$$\Rightarrow 0 < a_n < 1, \text{ if } n \geq N.$$

$$\Rightarrow 1 < 1 + a_n < 2, \text{ if } n \geq N$$

$$\Rightarrow \frac{a_n}{1+a_n} > \frac{1}{2}a_n, \text{ if } n \geq N$$

$$\Rightarrow \sum_{n=N}^{\infty} \frac{a_n}{1+a_n} > \sum_{n=N}^{\infty} \frac{1}{2}a_n.$$

Since $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=N}^{\infty} a_n$ also diverges.

By comparison test, then we have $\sum_{n=N}^{\infty} \frac{a_n}{1+a_n}$ also diverges.

Thus, we have $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} > \sum_{n=N}^{\infty} \frac{a_n}{1+a_n}$ diverges, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ also diverges.

This is an contradiction.

Therefore, $\sum \frac{a_n}{1+a_n}$ diverges. ∎

4. Rudin chapter 3, ※14. (a) (b)

If $\{S_n\}$ is a complex sequence, define its arithmetic means σ_n

by $\sigma_n = \dfrac{S_0+S_1+S_2+\cdots+S_n}{n+1}$ $n=0,1,2,\cdots\cdots$.

(a) If $\lim S_n = S$, prove that $\lim \sigma_n = S$.

(b) Construct a sequence $\{S_n\}$ which does not converge, although $\lim \sigma_n = 0$.

<pf>

(a)

Given $\varepsilon>0$, since $\lim\limits_{n\to\infty} S_n = S$, then $\exists N\in\mathbb{N}$ s.t. $|S_n - S| < \varepsilon/2$, if $n\geq N$.

$$|\sigma_n - S| = \left|\frac{S_0+S_1+\cdots+S_n}{n+1} - S\right| = \left|\frac{(S_0-S)+(S_1-S)+\cdots+(S_n-S)}{n+1}\right|$$

$$\leq \frac{|S_0-S|+|S_1-S|+\cdots+|S_n-S|}{n+1}$$

$$\Rightarrow |S_n - S| \leq \frac{|S_0-S|+|S_1-S|+\cdots+|S_{N-1}-S|}{n+1} + \frac{|S_N-S|+\cdots+|S_n-S|}{n+1}$$

For $\frac{\varepsilon}{2}>0$, by Archimedean principle, then $\exists N_0\in\mathbb{N}$ such that

$$N_0\cdot\frac{\varepsilon}{2} > |S_0-S|+|S_1-S|+\cdots+|S_{N-1}-S|.$$

$$\Rightarrow \frac{|S_0-S|+|S_1-S|+\cdots+|S_{N-1}-S|}{N_0} < \frac{\varepsilon}{2}.$$

Choose $N_1 = \max\{N, N_0\} \in \mathbb{N}$.

If $n \geq N_1$, then we have

$$|\sigma_n - s| = \left| \frac{s_0 + s_1 + \cdots + s_n}{n+1} - s \right|$$

$$\leq \frac{|s_0 - s| + |s_1 - s| + \cdots + |s_{N-1} - s|}{n+1} + \frac{|s_N - s| + \cdots + |s_n - s|}{n+1}$$

$$< \frac{|s_0 - s| + |s_1 - s| + \cdots + |s_{N-1} - s|}{N_0} + \underbrace{\frac{(n-N+1)}{n+1}}_{<1} \cdot \frac{\varepsilon}{2},$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $\lim_{n\to\infty} \sigma_n = s$.

(b)

Let $s_n = (-1)^n$.

$\sigma_0 = \frac{1}{1} = 1$, $\sigma_1 = \frac{1+(-1)}{2} = 0$, $\sigma_2 = \frac{1+(-1)+1}{3} = \frac{1}{3}$, $\sigma_3 = \frac{1+(-1)+1+(-1)}{3} = 0$,

$\sigma_4 = \frac{1+(-1)+1+(-1)+(1)}{5} = \frac{1}{5}$, ...

Then $\lim_{n\to\infty} \sigma_n = 0$, but $\{s_n\}$ does not converge.

5. Salas §12-3 ※12, 18, 23, 26, 28, 36

12. $\sum \frac{1}{k(k+1)(k+2)}$

<sol>

$a_k = \frac{1}{k(k+1)(k+2)}$ $\quad b_k = \frac{1}{k^3}$

$\because \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^3}{k^3+3k^2+2k} = 1 > 0$ and $\sum \frac{1}{k^3}$ converges, by limit comparison test,

$\therefore \sum \frac{1}{k(k+1)(k+2)}$ converges.

※

18. $\sum \frac{7k+2}{2k^5+7}$

<sol>

$a_k = \frac{7k+2}{2k^5+7}$ $\quad b_k = \frac{1}{k^4}$

$\because \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{7k^5+2k^4}{2k^5+7} = \frac{7}{2} > 0$ and $\sum \frac{1}{k^4}$ converges, by limit comparison test,

$\therefore \sum \frac{7k+2}{2k^5+7}$ converges.

※

23. $\sum \frac{1+2^k}{1+5^k}$

<sol>

$$a_k = \frac{1+2^k}{1+5^k} \qquad b_k = \left(\frac{2}{5}\right)^k$$

$$\because \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{1+2^k}{1+5^k}}{\frac{2^k}{5^k}} = \lim_{k\to\infty} \frac{5^k}{1+5^k} \times \frac{1+2^k}{2^k} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{5^k}} \times \frac{1+\frac{1}{2^k}}{1} = 1 > 0.$$

and $\sum \left(\frac{2}{5}\right)^k$ converges, by limit comparison test,

$\therefore \sum \frac{1+2^k}{1+5^k}$ converges. ※

26. $\sum \frac{2k+1}{\sqrt{k^3+1}}$

<sol>

$$a_k = \frac{2k+1}{\sqrt{k^3+1}} \qquad b_k = \frac{1}{\sqrt{k}}$$

$$\because \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt{k}\cdot(2k+1)}{\sqrt{k^3+1}} = \lim_{k\to\infty} \sqrt{\frac{(2k+1)^2\cdot k}{k^3+1}} = \sqrt{4} = 2 > 0$$

and $\sum \frac{1}{\sqrt{k}}$ diverges, by limit comparison test,

$\therefore \sum \frac{2k+1}{\sqrt{k^3+1}}$ diverges. ※

28. $\sum \frac{1}{\sqrt{2k(k+1)}}$

<sol>

$$a_k = \frac{1}{\sqrt{2k(k+1)}} \qquad b_k = \frac{1}{k}$$

$\because \lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k}{\sqrt{2k(k+1)}} = \lim_{k\to\infty} \sqrt{\frac{k}{2k+2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} > 0$ and $\sum \frac{1}{k}$ diverges,

by limit comparison test,

$\therefore \sum \frac{1}{\sqrt{2k(k+1)}}$ diverges

※

36. $\sum \frac{2k!}{(2k)!}$

<sol>

$$a_k = \frac{2k!}{(2k)!} = \frac{2\times 1\times 2\times 3\times \cdots \times k}{1\times 2\times 3\times \cdots \times k\times (k+1)\times \cdots (2k)!} = \frac{2}{(k+1)\times(k+2)\times\cdots\times(2k)} \leq \frac{1}{k^2} = b_k$$

$a_1 = \frac{2}{2} = 1 \leq b_1 = 1$, $a_2 = \frac{4}{4!} = \frac{1}{6} \leq b_2 = \frac{1}{4}$, $a_3 = \frac{12}{6!} \doteq \frac{2}{5!} \doteq \frac{1}{60} \leq \frac{1}{9} = b_3$,

$\cdots\cdots$

$\because 0 < a_k \leq b_k$ and $\sum \frac{1}{k^2}$ converges, by comparison test,

$\therefore \sum \frac{2k!}{(2k)!}$ converges.

※