HW5

1. Salas §12-3 ※ 49. 50.

49. Let $\sum a_k$ be a series with nonnegative terms.

(a) Show that if $\sum a_k$ converges, then $\sum a_k^2$ converges.

(b) Give an example where $\sum a_k^2$ converges and $\sum a_k$ converges;

give an example where $\sum a_k^2$ converges but $\sum a_k$ diverges.

(a)

<pf> Since $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Let $\varepsilon = 1 > 0$, $\exists N \in \mathbb{N}$, such that if $k \geq N$, then $0 \leq a_k < 1$.

If $k \geq N$, then we have $0 \leq a_k^2 < a_k < 1$.

$$\Rightarrow \sum_{k=N}^{\infty} a_k^2 \leq \sum_{k=N}^{\infty} a_k \leq \sum_{k=1}^{\infty} a_k < +\infty.$$

So $\sum_{k=1}^{\infty} a_k^2$ converges.

(b)

Ex: $a_k = \frac{(-1)^k}{k} \Rightarrow a_k^2 = \frac{1}{k^2}$

Then $\sum a_k$ converges and $\sum a_k^2$ converges. ※

Ex: $a_k = \frac{1}{k} \Rightarrow a_k^2 = \frac{1}{k^2}$

Then $\sum a_k$ diverges and $\sum \frac{1}{k^2}$ converges. ※

50. Let $\sum a_k$ be a series with nonnegative terms.

Show that if $\sum a_k^2$ converges, then $\sum \frac{a_k}{k}$ converges.

<pf>

Since $0 \le a_k$ for all $k \in \mathbb{N}$, then $0 \le (a_k - \frac{1}{k})^2 \le a_k^2 + \frac{1}{k^2}$ for all $k \in \mathbb{N}$.

Since $\sum a_k^2$ and $\sum \frac{1}{k^2}$ both converges, by comparison test, then $\sum_{k=1}^{\infty} (a_k - \frac{1}{k})^2$ converges.

Since $\sum_{k=1}^{\infty} (a_k - \frac{1}{k})^2 = \sum_{k=1}^{\infty} (a_k^2 - 2a_k \cdot \frac{1}{k} + \frac{1}{k^2}) = \sum_{k=1}^{\infty} a_k^2 - 2\sum_{k=1}^{\infty} a_k \cdot \frac{1}{k} + \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty$,

then $\sum_{k=1}^{\infty} \frac{a_k}{k}$ also converges. $\blacksquare$

2. Salas §12-4 ※ 1, 4, 12, 16, 22, 36, 40.

1. $\sum_{k=1}^{\infty} \frac{10^k}{k!}$ converges.

<sol> Let $a_k = \frac{10^k}{k!} > 0$.

Since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{10^{k+1}}{(k+1)!}}{\frac{10^k}{k!}} = \lim_{k\to\infty} \frac{10}{k+1} = 0 < 1$, by ratio test,

then $\sum_{k=1}^{\infty} \frac{10^k}{k!}$ converges.

※

4. $\sum_{k=1}^{\infty} \left(\frac{k}{2k+1}\right)^k$ converges.

<sol>

Let $a_k = \left(\frac{k}{2k+1}\right)^k > 0$

Since $\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2} < 1$, by root test,

then $\sum_{k=1}^{\infty} a_k$ converges.

※

12. $\sum_{k=1}^{\infty} \frac{2k+\sqrt{k}}{k^3+\sqrt{k}}$ converges.

<sol>

Let $a_k = \frac{2k+\sqrt{k}}{k^3+\sqrt{k}}$ and $b_k = \frac{1}{k^2}$.

$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{2k^3+k^{5/2}}{k^3+k^{5/2}} = \lim_{k\to\infty} \frac{2+\frac{1}{\sqrt{k}}}{1+\frac{1}{\sqrt{k}}} = \frac{2}{1} = 2 < \infty$, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges,

by limit comparison test,

then $\sum_{k=1}^{\infty} \frac{2k+\sqrt{k}}{k^3+\sqrt{k}}$ converges.

※

16. $\sum \frac{2^k \cdot k!}{k^k}$ converges.

<sol>

Let $a_k = \frac{2^k \cdot k!}{k^k} > 0$.

Since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2^{k+1}\cdot(k+1)!}{(k+1)^{k+1}} \times \frac{k^k}{2^k \cdot k!} = \lim_{k\to\infty} 2\cdot\left(\frac{k}{k+1}\right)^k$

$= \lim_{k\to\infty} \frac{2}{(1+\frac{1}{k})^k} = \frac{2}{e} < \infty$, by ratio test,

then $\sum_{k=1}^{\infty} \frac{2^k \cdot k!}{k^k}$ converges.

※

22.

$$\sum \frac{(k!)^2}{(2k)!} \text{ converges.}$$

<sol> Let $a_k = \dfrac{(k!)^2}{(2k)!} > 0.$

Since $\displaystyle\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{[(k+1)!]^2}{(2k+2)!} \times \frac{(2k)!}{(k!)^2} = \lim_{k\to\infty} \frac{(k+1)(k+1)}{(2k+1)(2k+2)} = \frac{1}{4} < 1,$

by ratio test, then $\displaystyle\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ converges.

36. $\displaystyle\sum_{k=1}^{\infty} (\sqrt{k} - \sqrt{k-1})^k$ converges.

<sol>

Let $a_k = (\sqrt{k} - \sqrt{k-1})^k > 0.$

Since $\displaystyle\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} (\sqrt{k} - \sqrt{k-1}) = \lim_{k\to\infty} \frac{1}{\sqrt{k}+\sqrt{k-1}} = 0 < 1,$

by root test, then $\displaystyle\sum_{k=1}^{\infty} (\sqrt{k} - \sqrt{k-1})^k$ converges.

40.

$$\frac{2}{3} + \frac{2\cdot 4}{3\cdot 7} + \frac{2\cdot 4\cdot 6}{3\cdot 7\cdot 11} + \frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 7\cdot 11\cdot 15} + \cdots\cdots$$ converges.

<sol>

Let $a_k = \dfrac{2\times 4\times 6\times \cdots \times (2k)}{3\times 7\times 11\times \cdots \times (4k-1)} > 0$.

Since $\displaystyle\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2\times 4\times \cdots \times (2k)\times (2k+2)}{3\times 7\times \cdots \times (4k-1)\times (4k+3)} \times \frac{3\times 7\times \cdots \times (4k-1)}{2\times 4\times \cdots \times (2k)}$

$$= \lim_{k\to\infty} \frac{2k+2}{4k+3}$$

$= \frac{1}{2} < 1$, by ratio test,

then $\displaystyle\sum_{k=1}^{\infty} a_k$ converges. ※

3. Rudin Ch3 ※12

(Hint: It might be useful to note that $r_n \to 0$ and $a_n = r_n - r_{n+1}$.)

Suppose $a_n > 0$ and Σa_n converges. Put $r_n = \sum_{m=n}^{\infty} a_m$.

(a)

Prove that $\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$, if $m < n$,

and deduce that $\Sigma \frac{a_n}{r_n}$ diverges.

(b)

Prove that $\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$

and deduce that $\Sigma \frac{a_n}{\sqrt{r_n}}$ converges.

<pf> (a)

Let $r_n = \sum_{m=n}^{\infty} a_m$. Then $r_n - r_{n+1} = \sum_{m=n}^{\infty} a_m - \sum_{m=n+1}^{\infty} a_m = a_n$ and $r_n > 0$.

Given $\varepsilon > 0$, since $\sum_{m=1}^{\infty} a_m$ converges, then $\exists N \in \mathbb{N}$ such that

if $n \geq N$, we have $|r_n| = \left| \sum_{m=1}^{n-1} a_m - \sum_{m=1}^{\infty} a_m \right| < \varepsilon$.

That is, $\lim_{n \to \infty} r_n = 0$.

Since $a_n > 0$, then $r_n - r_{n+1} > 0$, that is, $\{r_n\}$ is a decreasing sequence.

If $n>m$ and $a_n>0$, then $r_n<r_m$ and

$$\frac{a_m}{r_m}+\frac{a_{m+1}}{r_{m+1}}+\cdots+\frac{a_n}{r_n} \geqslant \frac{a_m}{r_m}+\frac{a_{m+1}}{r_m}+\cdots+\frac{a_n}{r_m}$$

$$=\frac{1}{r_m}(a_m+a_{m+1}+\cdots+a_n)$$

$$>\frac{1}{r_m}(a_m+a_{m+1}+\cdots+a_{n+1})$$

$$=\frac{1}{r_m}(r_m-r_n)=1-\frac{r_n}{r_m} \quad \text{— ①}$$

Suppose $\sum_{m=1}^{\infty}\frac{a_m}{r_m}$ converges. Given $\varepsilon>0$; $\exists N\in\mathbb{N}$, such that $\left|\sum_{m=N}^{\infty}\frac{a_m}{r_m}\right|<\varepsilon$.

Then $\varepsilon>\sum_{m=N}^{\infty}\frac{a_m}{r_m}\geqslant\sum_{m=N}^{n}\frac{a_m}{r_m}>1-\frac{r_n}{r_N}$, if $n\geq N$.

$\Rightarrow \frac{r_n}{r_N}>1-\varepsilon$, if $n\geq N$. $\Rightarrow \lim_{n\to\infty}\frac{r_n}{r_N}\geq 1-\varepsilon>0$. (this is a contradiction)

But we know that $\lim_{n\to\infty}r_n=0$, then $\lim_{n\to\infty}\frac{r_n}{r_N}=0$.

Thus, $\sum_{m=1}^{\infty}\frac{a_m}{r_m}$ diverges. ∎

(b)

$$\frac{a_n}{\sqrt{r_n}}=\frac{r_n-r_{n+1}}{\sqrt{r_n}}=\frac{(\sqrt{r_n}-\sqrt{r_{n+1}})(\sqrt{r_n}+\sqrt{r_{n+1}})}{\sqrt{r_n}}=(\sqrt{r_n}-\sqrt{r_{n+1}})\cdot\left(1+\frac{\sqrt{r_{n+1}}}{\sqrt{r_n}}\right)$$

Since $\{r_n\}$ is a decreasing sequence, then $r_{n+1}<r_n \Rightarrow \frac{\sqrt{r_{n+1}}}{\sqrt{r_n}}<1$.

So $\frac{a_n}{\sqrt{r_n}}<2(\sqrt{r_n}-\sqrt{r_{n+1}})$. — ②

Then $\sum_{n=1}^{\infty}\frac{a_n}{\sqrt{r_n}}<\sum_{n=1}^{\infty}2\cdot(\sqrt{r_n}-\sqrt{r_{n+1}})=2\sum_{n=1}^{\infty}(\sqrt{r_n}-\sqrt{r_{n+1}})$

$$=2\left[(\sqrt{r_1}-\sqrt{r_2})+(\sqrt{r_2}-\sqrt{r_3})+\cdots\right]=2\sqrt{r_1}<\infty.$$

So $\sum_{n=1}^{\infty}\frac{a_n}{\sqrt{r_n}}$ converges. ∎