HW 6

1. The function e^x has two identical definitions (show in class):

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n.$$

Without using differentiation, do the following:

(a) Show the series above converges absolutely for all x.

(b) Using either definition, show that for all $x, y \in \mathbb{R}$, $e^{x+y} = e^x \cdot e^y$.

(c) Show that for all $x \in \mathbb{R}$, $e^{-x} = \frac{1}{e^x}$, and therefore $e^{x-y} = \frac{e^x}{e^y}$.

(d) Show that $e^x > 0$ for all $x \in \mathbb{R}$

(e) Show that for all $x, y \in \mathbb{R}$, $(e^x)^y = e^{xy}$.

Note that you can't raise $1+\frac{x}{n}$ above to the power ny, since that is what we are proving here.

(f) Show that e^x is strictly increasing. That is, $x>y \Rightarrow e^x > e^y$.
(It might be useful to first show that $e^x > 1$ $\forall x>0$.)

Note that you are <u>Not</u> allowed to use any "power rule" you have learnt before. You need to derive them.

<pf> Define: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ — ① , $e^x = \lim_{n\to\infty} (1+\frac{x}{n})^n$ — ②

(a)

Let $a_k = \frac{x^k}{k!}$ $\forall k \in \mathbb{N}$.

Since $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \limsup_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} \right| = \limsup_{k\to\infty} \frac{|x|}{k+1} \to 0 < 1$ for all $x \in \mathbb{R}$,

by Ratio test, then $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely for all $x \in \mathbb{R}$. ■

(b) ① $\forall x, y \in \mathbb{R}$, $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$, $\sum_{l=0}^{\infty} \frac{y^l}{l!} = e^y$, $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely,

by <u>theorem 3.50 / Rudin</u>, then $\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!} \right) = e^x \cdot e^y$.

$$\begin{aligned}
\text{So, } e^x \cdot e^y &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{x^k}{k!} \cdot \frac{y^{n-k}}{(n-k)!} \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} C_k^n x^k y^{n-k} \right) \\
&= \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} \\
&= e^{x+y}.
\end{aligned}$$

② $\forall x, y \in \mathbb{R}$, $\lim_{n\to\infty} (1+\frac{x}{n})^n = e^x$, $\lim_{n\to\infty} (1+\frac{y}{n})^n = e^y$, $\lim_{n\to\infty} (1+\frac{x+y}{n})^n = e^{x+y}$.

Now, we have

$$\begin{aligned}
\left(1+\frac{x+y}{n}\right)^n = \left(\frac{n+x+y}{n}\right)^n &= \left(\frac{n+x}{n}\right)^n \cdot \left(\frac{n+y}{n}\right)^n \cdot \left(\frac{n(n+x+y)}{(n+x)(n+y)}\right)^n \\
&= \left(1+\frac{x}{n}\right)^n \cdot \left(1+\frac{y}{n}\right)^n \cdot \left(\frac{n(n+x+y)}{(n+x)(n+y)}\right)^n.
\end{aligned}$$

(b)

Claim: $\lim_{n\to\infty} \left(\frac{n(n+x+y)}{(n+x)(n+y)}\right)^n = 1.$ —(*)

If (*) hold, then $e^{x+y} = \lim_{n\to\infty} \left(1+\frac{x+y}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n \cdot \left(1+\frac{y}{n}\right)^n \cdot \left(\frac{n(n+x+y)}{(n+x)(n+y)}\right)^n$

$$= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \cdot \lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n \cdot \lim_{n\to\infty}\left(\frac{n(n+x+y)}{(n+x)(n+y)}\right)^n$$

$$= e^x \cdot e^y \cdot 1 = e^x \cdot e^y.$$

Now, $\frac{n(n+x+y)}{(n+x)(n+y)} = \frac{n^2+nx+ny}{n^2+nx+ny+xy} = 1 - \frac{xy}{n^2+nx+ny+xy} = 1 - \frac{xy}{(n+x)(n+y)}.$

$$\Rightarrow \left(\frac{n(n+x+y)}{(n+x)(n+y)}\right)^n = \left[1 - \frac{xy}{(n+x)(n+y)}\right]^n$$

$$= 1 - \frac{nxy}{(n+x)(n+y)} + \sum_{k=2}^{n} C_k^n \cdot (-1)^k \cdot \left(\frac{xy}{(n+x)(n+y)}\right)^k$$

$$= 1 - \underbrace{\frac{nxy}{(n+x)(n+y)}}_{\to 0,\ \text{as } n\to\infty} + \sum_{k=2}^{n} (-1)^k \cdot \frac{1}{k!} \cdot (xy)^k \cdot \underbrace{\frac{(n-k+1)(n-k+2)\cdots\cdots(n-1)\cdot n}{(n+x)^k (n+y)^k}}_{\to 0,\ \text{as } n\to\infty}$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{n(n+x+y)}{(n+x)(n+y)}\right)^n = 1.$$

Thus, by definition ①, definition ②, then we have $e^x \cdot e^y = e^{x+y}$.

■

(c)

By def ①, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!}$, then we have $e^0 = 1 + \sum_{k=1}^{\infty} \frac{0^k}{k!} = 1$.

By def ②, $e^x = \lim_{n\to\infty} (1+\frac{x}{n})^n$, then we have $e^0 = \lim_{n\to\infty} (1+\frac{0}{n})^n = 1$.

$\forall x \in \mathbb{R}$, by (b), then $e^0 = e^{x+(-x)} = e^x \cdot e^{-x} = 1$.

So $e^{-x} = \frac{1}{e^x}$ for all $x \in \mathbb{R}$.

Thus, $e^{x-y} = e^{x+(-y)} \underset{(\because (b) \text{ hold})}{=} e^x \cdot e^{-y} = e^x \cdot \frac{1}{e^y} = \frac{e^x}{e^y}$ $\blacksquare$

(d)

$\forall x > 0$, by definition ①, then $\frac{x^k}{k!} > 0$ for all $k \in \mathbb{N}$.

Thus, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \sum_{k=0}^{\infty} \frac{x^k}{k!} > 1 > 0$, as $x > 0$.

$\forall x > 0$, by definition ②, then $(1+\frac{x}{n})^n > 1$ for all $n \in \mathbb{N}$.

Thus, $e^x = \lim_{n\to\infty} (1+\frac{x}{n})^n \geq 1 > 0$, as $x > 0$.

By (c), then $e^{-x} = \frac{1}{e^x} > 0$, as $x > 0$.

Therefore, we have $e^x > 0$ for all $x \in \mathbb{R}$. $\blacksquare$

(e)

Define: $a^x = e^{x \log a}$, $a>0$, for all $x \in \mathbb{R}$.

Then $(e^x)^y = e^{y \log e^x} = e^{xy}$ for any $x, y \in \mathbb{R}$. ■

(f)

By definition ①, definition ②, then we have $e^x > 1$, as $x > 0$. — (*)

If $x > y$, then $x - y > 0$.

By (*), then we have $e^{x-y} > 1$.

By (c), then $e^{x-y} = \frac{e^x}{e^y} > 1$.

Since $e^y > 0$, then $e^x > e^y$.

So, e^x is strictly increasing. ■

2. Salas.

§12-5 ※ 10, 18, 20, 35, 42

10.
$$\sum (-1)^k \cdot \frac{(k!)^2}{(2k)!}$$

<sol>

Let $a_k = \frac{(k!)^2}{(2k)!} > 0$.

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{((k+1)!)^2}{(2k+2)!} \times \frac{(2k)!}{(k!)^2} = \lim_{k\to\infty} \frac{(k+1)^2}{(2k+1)(2k+2)} = \frac{1}{4} < 1,$$

by Ratio test, then $\sum (-1)^k \cdot \frac{(k!)^2}{(2k)!}$ is absolutely convergent.

※

18.
$$\sum \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)$$

<sol>

Let $a_k = \left| \frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right| = \left| \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\cdot\sqrt{k+1}} \right| = \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\cdot\sqrt{k+1}} = \frac{1}{\sqrt{k}\sqrt{k+1}(\sqrt{k+1}+\sqrt{k})} > 0.$

Then $a_k \leq \frac{1}{(\sqrt{k})^3} = \frac{1}{k^{3/2}}$ $\forall k \in \mathbb{N}$.

Since $\sum \frac{1}{k^{3/2}}$ converges ($\because$ p-series) and $\sum a_k \leq \sum \frac{1}{k^{3/2}}$, by comparison test,

then $\sum a_k$ converges.

So, $\sum \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)$ is absolutely convergent.

※

20.

$$\sum (-1)^k \cdot \frac{k+2}{k^2+k}$$

<sol>

Let $a_k = \left| (-1)^k \cdot \frac{k+2}{k^2+k} \right| = \frac{k+2}{k^2+k} > 0.$

Then $a_k > \frac{k}{2k^2} = \frac{1}{2k} \quad \forall k \in \mathbb{N}.$

Since $\sum \frac{1}{2k}$ diverges and $\sum a_k > \sum \frac{1}{2k}$, by comparison test,

then $\sum a_k$ diverges.

Let $b_k = \frac{k+2}{k^2+k} > 0$. Since $\lim_{k\to\infty} b_k = \lim_{k\to\infty} \frac{k+2}{k^2+k} = \frac{0}{1} = 0$, $b_k > 0$,

then $\sum (-1)^k \cdot b_k$ converges.

Thus, it is conditional convergence. ※

35.

$$\sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{1}{k^3}$$

<sol>

Let $S = \sum_{k=1}^{\infty} (-1)^{k+1} \cdot \frac{1}{k^3}$ and $S_9 = \sum_{k=1}^{9} (-1)^{k+1} \cdot \frac{1}{k^3}.$

Then $|S - S_9| = \left| \sum_{k=10}^{\infty} (-1)^{k+1} \cdot \frac{1}{k^3} \right|$

$$= \left| -\frac{1}{10^3} + \frac{1}{11^3} - \frac{1}{12^3} + \frac{1}{13^3} - \frac{1}{14^3} + \cdots \right|$$

$$= \left| \frac{1}{10^3} - \underbrace{\left(\frac{1}{11^3} - \frac{1}{12^3}\right)}_{>0} + \underbrace{\left(\frac{1}{13^3} - \frac{1}{14^3}\right)}_{>0} + \cdots \right|$$

$$< \frac{1}{10^3} = \frac{1}{1000}$$

※

3. Salas §12-8 ※ 8, 20, 30, 35.

§12-5 ※ 42,

Let $a_{k+1} \leq a_k$ and $a_k > 0$ $\forall k \in \mathbb{N}$. with $\lim_{k\to\infty} a_k = 0$.

Does $\sum_{k=1}^{\infty} (-1)^k a_k$ converges? Answer: Yes

<pf>

Let $S_{2m} = \sum_{k=1}^{2m} (-1)^k \cdot a_k$, $\forall m \in \mathbb{N}$.

Since $S_{2m+2} \leq S_{2m}$ and $S_{2m} > 0$ $\forall m \in \mathbb{N}$, then $\lim_{m\to\infty} S_{2m} = \inf_m \{S_{2m}\}$.

Let $S_{2m+1} = \sum_{k=1}^{2m+1} (-1)^k \cdot a_k$, $\forall m \in \mathbb{N}$.

Then $S_{2m+1} = S_{2m} - a_{2m+1}$, $\forall m \in \mathbb{N}$.

Since $\lim_{k\to\infty} a_k = 0$, then $\lim_{m\to\infty} a_{2m+1} = 0$.

Since $S_{2m+1} = S_{2m} - a_{2m+1}$, $\lim_{m\to\infty} S_{2m} = \inf_m \{S_{2m}\}$, $\lim_{m\to\infty} a_{2m+1} = 0$,

then $\lim_{m\to\infty} S_{2m+1} = \inf_m \{S_{2m}\}$.

Since $\lim_{m\to\infty} S_{2m} = \lim_{m\to\infty} S_{2m+1} = \inf_m \{S_{2m}\}$, let $S_n = \sum_{k=1}^{n} (-1)^k \cdot a_k$, $\forall n \in \mathbb{N}$,

then $\lim_{n\to\infty} S_n = \inf_m \{S_{2m}\}$. (converges !!)

∎

§ 12-8

※ 8.

$$\sum \frac{(-1)^k}{\sqrt{k}} x^k$$

<sol>

Let $a_k = \frac{(-1)^k}{\sqrt{k}} \cdot x^k$.

If $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \limsup_{k\to\infty} \sqrt[k]{\frac{|x|^k}{\sqrt{k}}} = \limsup_{k\to\infty} \frac{|x|}{\sqrt[k]{\sqrt{k}}} = |x| < 1$,

then $\sum a_k$ converges, by Root test,

At $x=1$, then $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$ converges, since $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$.

At $x=-1$, then $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges, since p-series.

So, $(-1, 1]$

※

20. $\sum \frac{7^k}{k!} x^k$

<sol>

Let $a_k = \frac{7^k}{k!} x^k$.

If $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \limsup_{k\to\infty} \left| \frac{7^{k+1} \cdot x^{k+1}}{(k+1)!} \times \frac{k!}{7^k x^k} \right| = \limsup_{k\to\infty} \left|\frac{7x}{k+1}\right| = 0 < 1$,

then $\sum a_k$ converges, by Ratio test,

So, $(-\infty, \infty)$ ※

30. $\sum \frac{k^3}{e^k} (x-4)^k$

<sol>

Let $a_k = \frac{k^3}{e^k} \cdot (x-4)^k$.

If $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \limsup_{k\to\infty} \sqrt[k]{\frac{k^3}{e^k} |x-4|^k} = \limsup_{k\to\infty} \frac{(\sqrt[k]{k})^3}{e} \cdot |x-4| = \frac{|x-4|}{e} < 1$,

then $\sum a_k$ converges, by Root test.

At $x = 4+e$,

then $\sum_{k=1}^{\infty} \frac{k^3}{e^k} \cdot e^k = \sum_{k=1}^{\infty} k^3$ diverges.

At $x = 4-e$,

then $\sum_{k=1}^{\infty} \frac{k^3}{e^k} (-e)^k = \sum_{k=1}^{\infty} (-1)^k \cdot k^3$ diverges,

since $\lim_{k\to\infty} k^3 \neq 0$.

So, $(4-e, 4+e)$ ※

35.

$$\sum (-1)^k \cdot \left(\frac{2}{3}\right)^k \cdot (x+1)^k$$

<sol>

Let $a_k = (-1)^k \cdot \left(\frac{2}{3}\right)^k \cdot (x+1)^k$

If

$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \limsup_{k\to\infty} \left(\frac{2}{3}|x+1|\right) = \frac{2}{3}|x+1| < 1$, then $\sum a_k$ converges, by Root test.

$\frac{2}{3}|x+1| < 1 \Rightarrow |x+1| < \frac{3}{2} \Rightarrow \frac{-3}{2} < x+1 < \frac{3}{2} \Rightarrow \frac{-5}{2} < x < \frac{1}{2}$

At $x = \frac{-5}{2}$, then $\sum_{k=1}^{\infty} (-1)^k \cdot \left(\frac{2}{3}\right)^k \left(\frac{-3}{2}\right)^k = \sum_{k=1}^{\infty} 1$ diverges.

At $x = \frac{1}{2}$, then $\sum_{k=1}^{\infty} (-1)^k \cdot \left(\frac{2}{3}\right)^k \cdot \left(\frac{3}{2}\right)^k = \sum_{k=1}^{\infty} (-1)^k$ diverges.

So, $\left(\frac{-5}{2}, \frac{1}{2}\right)$

※