1. (a) Assume $x \in E^{\circ}$

$\exists \delta > 0$ s.t. $(x-\delta, x+\delta) \subseteq E$

$\because (x-\delta, x+\delta)$ is open

So, $\forall x' \in (x-\delta, x+\delta)$

$\exists r > 0$ s.t. $(x'-r, x'+r) \subseteq (x-\delta, x+\delta) \subseteq E$

$\Rightarrow (x'-r, x'+r) \subseteq E$

$\Rightarrow x' \in E^{\circ}$ $\Leftarrow$ x' is an interior point

Every point in $(x-\delta, x+\delta)$ is an interior point in E

$\Rightarrow (x-\delta, x+\delta) \subseteq E^{\circ}$

$\Rightarrow E^{\circ}$ is open

(b) Prove that E is open $\Leftrightarrow E^{\circ} = E$

($\Rightarrow$) $E = E^{\circ} \Leftrightarrow E \subseteq E^{\circ}$ and $E^{\circ} \subseteq E$

E is open, that is, every point of E is an interior point $\Rightarrow E \subseteq E^{\circ}$

E° is the set of interior points of $E \Rightarrow E^{\circ} \subseteq E$

$\therefore E = E^{\circ}$

($\Leftarrow$) $E^{\circ} = E$

$\because E^{\circ}$ is open $\therefore E$ is open

(c) G is open

$\Rightarrow \forall y \in G\ \exists t > 0$ s.t. $(y-t, y+t) \subseteq G \subseteq E$

that is, every point of G is an interior point of E

$\Rightarrow G \subseteq E^{\circ}$

E is an interior point

$\in \Rightarrow E^\circ \subseteq E$ $\Rightarrow E \subseteq E^\circ$

open

(c) G is open

$\Rightarrow \forall y \in G \ \exists t > 0$ s.t. $(y-t, y+t) \subset G \subset E$

that is, every point of G is an interior point of E

$\Rightarrow G \subseteq E^\circ$

(d) $E^\circ = \bigcup\limits_{G \subset E,\, G \text{ is open}} G$

1. $\supset$, by (c)

2. $\subset$, by (a)

(a) Prove that $(0,1) \subset \mathbb{R}$ is not compact

$\Rightarrow \exists$ a open cover that doesn't have finite subcover

$(0,1) \subset \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1)$

Assume G has a finite subcover G'

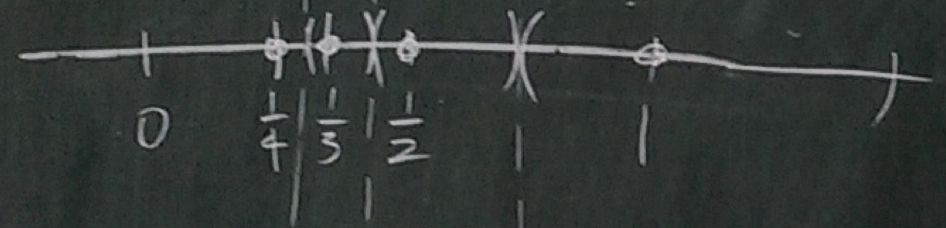
Since G' is a finite subcover there exists a largest n s.t. $(\frac{1}{n}, 1) \in G'$

However $(\frac{1}{n+1}, 1) \notin G'$

Thus, G' doesn't cover $(0,1)$

(b) Prove that $\{\frac{1}{n}\}_{n=1}^{\infty} \subset \mathbb{R}$ is not compact

$\Rightarrow \exists$ a open cover that doesn't have finite subcover



$\{\frac{1}{n}\}_{n=1}^{\infty} \subset \bigcup_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right), \frac{1}{n} + \frac{1}{n}\left(\frac{1}{n-1} - \frac{1}{n}\right) \right) \cup (\frac{3}{4}, 2)$

$= \bigcup_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{2n(n+1)}, \frac{1}{n} + \frac{1}{2n(n-1)} \right) \cup (\frac{3}{4}, 2)$

$\because$ every cover contains only one point,

$\therefore$ it doesn't have finite subcover

$\Rightarrow \{\frac{1}{n}\}_{n=1}^{\infty}$ is not compact

(c) Prove $\{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty}$ is compact

any open cover of $(\{0\} \cup \{\frac{1}{n}\}_{n=1}^{\infty})$ must contain the point $\{0\}$

In $\mathbb{R}$, the form of open cover is (a,b) $a<b$

thus, there exist a subcover of open cover s.t. $0 \in (a,b)$, that is, $a<0<b$

By the Archimeden Property

$(\forall \varepsilon > 0, \exists n \in \mathbb{N}$ st $0 < \frac{1}{n} < \varepsilon)$

$\forall b > 0, \exists m \in \mathbb{N}$ s.t. $0 < \frac{1}{m} < b$

$a < 0 < \frac{1}{m} < b$.

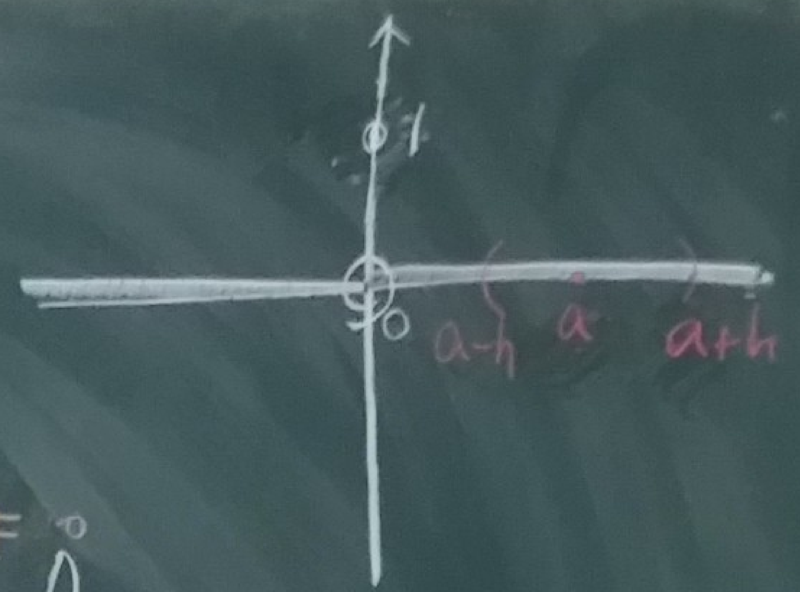
(a,b) contains $\{0\}$ $\{\frac{1}{n}\}_{n=m}^{\infty}$

$\{\frac{1}{n}\}_{n=1}^{n=m-1}$ has finite points

$\therefore$ it can be covered by finite open set

thus any open cover has a finite subcover QED

3. Set $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$



At $x = 0$

h很近0 $\neq 0$

$$\lim_{h \to 0} [f(0+h) - f(0-h)] = \lim_{h \to 0} [0 - 0] = 0$$

$\therefore f(x)$ satisfies the statement, but $f(x)$ isn't continuous at $x = 0$

$$\lim_{x \to 0} f(x) = 0 \neq 1 = f(0)$$

At $x = a, a \neq 0$

$$\lim_{h \to 0} [f(a+h) - f(a-h)]$$

$$= \lim_{h \to 0} [0 - 0]$$

$$= 0$$

令 $h = \frac{|a|}{2} > 0$

s.t. $f(a+h) = f(a-h)$

$= 0$

4. $f: X \to Y$

$C \subset Y$ is closed, $f^{-1}(C)$ is closed in X

$\Leftrightarrow$ $E \subset Y$ is open, $f^{-1}(E)$ is open in X

$\Leftrightarrow$ f is continuous

Lemma: $f: X \to Y$, $f^{-1}(Y^c) = [f^{-1}(Y)]^c$

Case1. $\forall x \in f^{-1}(Y^c)$

since $f^{-1}(Y^c) := \{x \in X \mid f(x) \in Y^c\}$

$\Rightarrow f(x) \in Y^c$

$\Rightarrow f(x) \notin Y$

$\Rightarrow x \notin f^{-1}(Y)$

$\Rightarrow x \in [f^{-1}(Y)]^c$

Case2. $\forall x \in [f^{-1}(Y)]^c$

$\Rightarrow x \notin f^{-1}(Y)$

$\Rightarrow f(x) \notin Y$

$\Rightarrow f(x) \in Y^c$

$\Rightarrow x \in f^{-1}(Y^c)$

$\Rightarrow$ Assume that $f^{-1}(C)$ is closed for every closed subset $C \subset \mathbb{R}$

Let E be open on $\mathbb{R}$, then E^c is closed, so $\underbrace{f^{-1}(E^c) = [f^{-1}(E)]^c}_{\text{by Lemma}}$ is closed, so $f^{-1}(E)$ is open

$\Leftarrow$ Assume that $f^{-1}(E)$ is open for every open subset E on $\mathbb{R}$

Let C be closed in $\mathbb{R}$ then C^c is open.

so $\underbrace{f^{-1}(C^c) = (f^{-1}(C))^c}_{\text{by lemma}}$ is open so $f^{-1}(C)$ is closed

5. $\forall \beta \in \{a\}^c$

① $\beta > a$ $\because 1 \in \mathbb{R}$ and $\beta, a \in \mathbb{R}$

$\Rightarrow \beta - a \in \mathbb{R} \Rightarrow \frac{1}{\beta - a} \in \mathbb{R}$

$\therefore \exists r \in \mathbb{Q} \Rightarrow \frac{1}{r} \in \mathbb{Q}$

s.t. $\frac{1}{r} \cdot 1 > \frac{1}{\beta - a}$ (by Archimedean prop)

$\Rightarrow \frac{1}{r} > \frac{1}{\beta - a} \Rightarrow r < \beta - a \Rightarrow \underline{a < \beta - r}$

$\Rightarrow (\beta - r, \beta + r) \subset \{a\}^c$

② $\beta < a$

$\because 1 \in \mathbb{R}$ and $\beta, a \in \mathbb{R}$

$\Rightarrow a - \beta \in \mathbb{R} \Rightarrow \frac{1}{a - \beta} \in \mathbb{R}$

$\therefore \exists r \in \mathbb{Q} \Rightarrow \frac{1}{r} \in \mathbb{Q}$

s.t. $\frac{1}{r} \cdot 1 > \frac{1}{a - \beta}$ (by Archimedean property)

$\Rightarrow \frac{1}{r} > \frac{1}{a - \beta} \Rightarrow r < a - \beta$

$\Rightarrow a > \beta + r$

$\Rightarrow (\beta - r, \beta + r) \subset \{a\}^c$

$\boxed{\begin{array}{l}\therefore \forall \beta \in \{a\}^c \text{ is a interior point} \\ \Rightarrow \{a\}^c \text{ open} \Rightarrow \{a\} \text{ closed } \#\end{array}}$

$\beta \in \{a\}^c$

$|\beta - a| > 0$

$\exists r \in (0, |\beta - a|)$

$\Rightarrow a \notin (\beta - r, \beta + r) \Rightarrow (\beta - r, \beta + r) \subset \{a\}^c$

We've known:

① $f^{-1}(V) := \{x \in E \mid f(x) \in V\}$

② $\{a\} \subset \mathbb{R}$ is closed $\forall a \in \mathbb{R}$

③ $f^{-1}(V)$ is closed in $\mathbb{R}$ $\forall$ closed subset V

④ $Z(f) = \{p \in \mathbb{R} \mid f(p) = 0\}$

$Z(f) = \underline{\{p \in \mathbb{R} \mid f(p) = 0\}}$

$= \{p \in \mathbb{R} \mid f(p) \in \{0\}\} = f^{-1}(\{0\})$

by ②, $\{0\}$ is closed

$\Rightarrow$ by ③, $f^{-1}(\{0\})$ closed $\Rightarrow Z(f)$ closed $*$

6. Given $f:[0,1]\longrightarrow \mathbb{R}$ continuous, and suppose that $f([0,1])\subset \mathbb{Q}$. Prove that if $f(\frac{1}{2})=0$, then f is a constant function.

Case 1

$a,b\in Q$, $b-a>0$

$\exists k\in Q$ st $k\pi(b-a)>1$

$b>a+\frac{1}{k\pi}>a$

$a\in Q \wedge \frac{1}{k\pi}\in Q' \Rightarrow a+\frac{1}{k\pi}\in Q'$

Case 2

$a,b\in Q'$, $b-a>0$

$\exists k\in Q$ s.t. $k(b-a)>1$

$b-a>\frac{1}{k}$

$b>\frac{1}{k}+a>a$

Case 3

$a\in Q'$

$b\in Q$

$b-a>0$

$\exists k\in \mathbb{N}$ s.t.

$k(b-a)>1$

$b>a+\frac{1}{k}>a$

$\because$ The range of $f(x)$ can't contain irrational

$\therefore$ The range of $f(x)$ isn't a interval (a,b) $\forall a\neq b$

and $f(x)$ is continuous

$\Rightarrow$ The range of $f(x)$ is a constant number

$\Rightarrow$ $f(x)$ is a constant function