HW 7

1. Rudin Ch2, Problem 9 (a), (b), (c) and the following statement: $E^\circ = \bigcup_{\substack{G \subset E \\ G:\text{open}}} G$.

Let E° denote the set of all interior point of a set E.

[see Def 2.18 (e); E° is called the interior of E.]

(a) Prove that E° is always open

(b) Prove that E is open if and only if $E^\circ = E$

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

(*) $E^\circ = \bigcup_{\substack{G \subset E \\ G:\text{open}}} G$; E° is called the largest open subset contained in E.

<pf> (a)

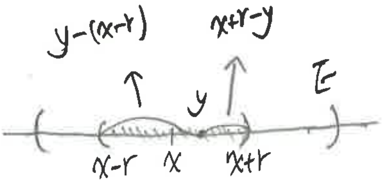
Let $E^\circ = \{$ all interior points of $E\}$.

$\forall x \in E^\circ$, $\exists r > 0$, then $x \in (x-r, x+r) \subset E$.

Let $B = (x-r, x+r)$.

Claim that $B \subset E^\circ$, that is, x is an interior point of E°.

$\forall y \in B$, $\exists r' = \frac{1}{2}\min\{x+r-y, y-(x-r)\} > 0$,



then $y \in (y-r', y+r') \subset B \subset E$.

So y is an interior point of E. Then $y \in E^\circ \Rightarrow B \subset E^\circ$

Therefore, E° is open set. ∎

(b) ① Assume $E^\circ = E$. By (a), since E° is open, then E is open.

② Assume E is open.

By definition of open set, then $E = \{\text{all interior points of } E\} = E^\circ$. ▨

(c) $\forall x \in G$, since G is open set, then $\exists r > 0$, such that $x \in (x-r, x+r) \subset G$.

Since $G \subset E$, then $x \in (x-r, x+r) \subset G \subset E$.

So x is an interior point of E.

Then $x \in E^\circ$, that is, $G \subset E^\circ$ ▨

(*) claim : $E^\circ = \bigcup_{\substack{G \subset E \\ G = \text{open}}} G$.

①

By (a), since E° is open set, $E^\circ \subset E$, then $E^\circ \subset \bigcup_{\substack{G \subset E \\ G = \text{open}}} G$.

②

By (c), if $G \subset E$, G is open, then $G \subset E^\circ$

$$\Rightarrow \bigcup_{\substack{G \subset E \\ G = \text{open}}} G \subset \bigcup E^\circ = E^\circ$$

▨

2.

Prove that (a) $(0,1) \subset \mathbb{R}$ is not compact (Note: open $\neq$ not closed)

(b) $\{\frac{1}{n}\}_{n=1}^{\infty} \subset \mathbb{R}$ is not compact

(c) $\{\frac{1}{n}\}_{n=1}^{\infty} \cup \{0\} \subset \mathbb{R}$ is compact.

It is called the compactification of (b).

<pf>

(a) $(0,1) = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1)$ Let $F = \{ (\frac{1}{n}, 1) \mid n \in \mathbb{N} \}$.

Then F is a open cover of $(0,1)$.

We choose a family F_1 of some intervals of F.

Let $F_1 = \{ (\frac{1}{n_1}, 1), (\frac{1}{n_2}, 1), \cdots, (\frac{1}{n_k}, 1) \}$.

Choose $N = \max\{n_1, n_2, \cdots, n_k\}$. Then $(\frac{1}{N}, 1) \supset (\frac{1}{n_i}, 1)$ for all $1 \le i \le k$.

Since $0 < \frac{1}{N}$, then we can choose a real number $x \in (0, \frac{1}{N})$ such that

$x \notin (\frac{1}{N}, 1)$, that is, $x \notin (\frac{1}{n_i}, 1)$ for all $1 \le i \le k$.

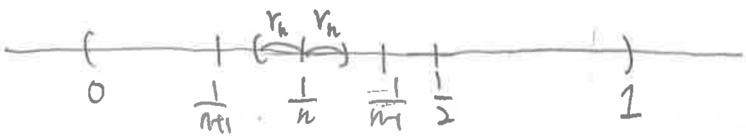
Thus, F_1 can not cover $(0,1)$. So $(0,1)$ is not compact. ∎

(b)

For each $\frac{1}{n}$, $n \geq 2$, let $r_n = \frac{1}{2} \min\left\{ \frac{1}{n-1} - \frac{1}{n}, \frac{1}{n} - \frac{1}{n+1} \right\} > 0$, $n \geq 2$,

then $\left(\frac{1}{n} - r_n, \frac{1}{n} + r_n\right) \cap \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}$, as $n \geq 2$.

As $n=1$, for point $\frac{1}{1}$, let $r_1 = \frac{1}{4} > 0$, then $\left(\frac{1}{1} - \frac{1}{4}, \frac{1}{1} + \frac{1}{4}\right) \cap \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \{1\}$.



Let $F = \left\{ \left(\frac{1}{n} - r_n, \frac{1}{n} + r_n\right) \middle| n \in \mathbb{N} \right\}$.

Then F is a open cover of $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, but its finite subcover can not cover $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

So $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is not compact. ∎

(c)

Let $E = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \cup \{0\}$.

$\forall x \in E$, then $0 \leq x \leq 1$. So E is bounded.

Given $\varepsilon > 0$, $\exists n \in \mathbb{N}$, s.t. $0 < \underbrace{\frac{1}{n}}_{\in E} < \varepsilon$. So 0 is a limit point of E.

By (b), we know that $\frac{1}{n} \in E$ is an isolated point. So 0 is the only limit point of E. Since $0 \in E$, then E is closed.

Since E is bounded, closed, by Heine-Borel theorem, then E is compact. ∎

3. Rudin Ch4, #1

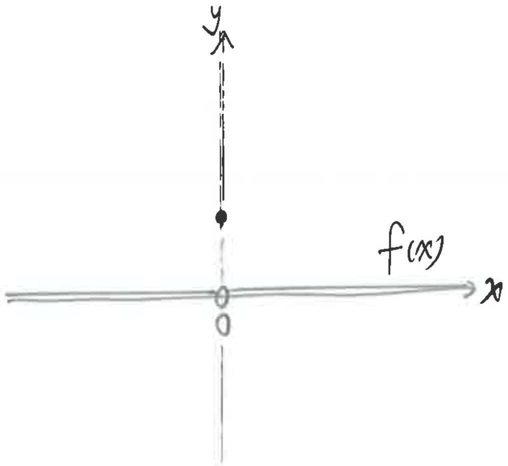
Suppose f is a real function defined on $\mathbb{R}^1$ which satisfies

$$\lim_{h\to 0}[f(x+h)-f(x-h)]=0 \text{ for every } x\in\mathbb{R}^1.$$

Does this imply that f is continuous?

<sol>

Answer: No!!



Example:

$$f(x)=\begin{cases}1, & x=0\\ 0, & x\neq 0.\end{cases}$$

Given $\varepsilon>0$.

① As $x\neq 0$, we have $\delta=\frac{|x|}{2}>0$.

If $0<|h|<\delta$, then $|f(x+h)-f(x-h)|=|0-0|=0<\varepsilon$.

$$\Rightarrow \lim_{h\to 0}[f(x+h)-f(x-h)]=0.$$

② As $x=0$, we choose $\delta=\varepsilon>0$.

If $0<|h|<\delta$, then $|f(0+h)-f(0-h)|=|0-0|=0<\varepsilon$.

$$\Rightarrow \lim_{h\to 0}[f(0+h)-f(0-h)]=0.$$

So f satisfies $\lim_{h\to 0}[f(x+h)-f(x-h)]=0$, $\forall x\in\mathbb{R}^1$, but f is not continuous at $x=0$.

4.

Prove that a continuous map $f: \mathbb{R} \to \mathbb{R}$ is continuous if and only if for all closed subset $C \subset \mathbb{R}$, $f^{-1}(C)$ is closed in $\mathbb{R}$.

<pf> ① Assume f is continuous on $\mathbb{R}$.

Let C be a closed subset of $\mathbb{R}$ and x be a limit point of $f^{-1}(C)$.

Claim: $x \in f^{-1}(C) = \{p \in C \mid f(p) \in C\}$, that is, $f(x) \in C$.

Given $\varepsilon > 0$, since f is continuous at x, then $\exists \delta > 0$ such that

if $p \in (x-\delta, x+\delta)$, we have $f(p) \in (f(x)-\varepsilon, f(x)+\varepsilon)$. —— (1)

For this $\delta > 0$, since x is a limit point of $f^{-1}(C)$, then $\exists q \neq x$,

$q \in (x-\delta, x+\delta)$ such that $q \in f^{-1}(C)$. ($\Rightarrow f(q) \in C$)

By (1), since $q \in (x-\delta, x+\delta)$, then $f(q) \in (f(x)-\varepsilon, f(x)+\varepsilon)$.

case 1: if $f(x) = f(q)$ and $f(q) \in C$, then $f(x) \in C$.

case 2: if $f(x) \neq f(q)$, then $f(x)$ is a limit point of C.

Since C is closed, then $f(x) \in C$.

Thus, $x \in f^{-1}(C)$. So $f^{-1}(C)$ is closed set.

(乙) Assume $f^{-1}(C)$ is closed in $\mathbb{R}$ for any C is closed.

Claim: f is continuous on $\mathbb{R}$, that is, f is continuous at x for any $x \in \mathbb{R}$.

, that is, for any $x \in \mathbb{R}$, given $\varepsilon > 0$, $\exists \delta > 0$, such that

if $y \in (x-\delta, x+\delta)$, we have $f(y) \in (f(x)-\varepsilon, f(x)+\varepsilon)$.

For any given $x \in \mathbb{R}$.

Given $\varepsilon > 0$, $(f(x)-\varepsilon, f(x)+\varepsilon)$ is open set, then $\mathbb{R} \setminus (f(x)-\varepsilon, f(x)+\varepsilon)$ is a closed set.

$$
\begin{aligned}
\text{Then } f^{-1}(\mathbb{R} \setminus (f(x)-\varepsilon, f(x)+\varepsilon)) &= f^{-1}(\mathbb{R}) \setminus f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \\
&= \mathbb{R} \setminus f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \text{ is closed.} \\
&\Rightarrow f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)) \text{ is open set}
\end{aligned}
$$

Since $f(x) \in (f(x)-\varepsilon, f(x)+\varepsilon)$, then $x \in f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon))$.

Since $f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon))$ is open set, then $\exists \delta > 0$ such that

$$(x-\delta, x+\delta) \subset f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon)).$$

If $y \in (x-\delta, x+\delta)$, then $y \in f^{-1}((f(x)-\varepsilon, f(x)+\varepsilon))$.

$$\Rightarrow f(y) \in (f(x)-\varepsilon, f(x)+\varepsilon).$$

So f is continuous at x for any $x \in \mathbb{R}$. ∎

5.

Rudin Ch4 ※3.

Let f be a continuous real function on a metric space $X = \mathbb{R}$.

Let $Z(f)$ be the set of all $p \in X$ at which $f(p) = 0$.

(the zero set of f)

Prove that $Z(f)$ is closed.

<pf> $Z(f) = f^{-1}(\{0\}) = \{ p \in \mathbb{R} \mid f(p) = 0 \}$.

Since $(-\infty, 0) \cup (0, \infty)$ is open set, then $\{0\} = \mathbb{R} \setminus ((-\infty, 0) \cup (0, \infty))$

is closed set.

Since f is continuous on $\mathbb{R}$, $\{0\}$ is closed set,

then $Z(f) = f^{-1}(\{0\})$ is closed set. ▨

6. Given $f:[0,1]\to\mathbb{R}$ continuous, and suppose that $f([0,1])\subset\mathbb{Q}$.

Prove that if $f(\frac{1}{2})=0$, then f is a constant function.

<pf>

Suppose there is a $x\in[0,1]$ such that $f(x)>0$.

By density of real number, then there is a $r\in\mathbb{Q}^c$ between

0 and $f(x)$.

Since $f(\frac{1}{2})=0$, $f(x)>0$, $0<r<f(x)$, f is continuous on $[0,1]$,

by I.V.T., then $\exists\, x_0\in(0,1)$ such that $f(x_0)=r$.

Since $f([0,1])\subset\mathbb{Q}$, but $r\in\mathbb{Q}^c$, so this is a contradiction. ∎

7. Salas 2.4 ※ 6, 14, 26, 29, 34

6. $f(x) = |4 - x^2|$, $x = 2$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 - 4) = 0$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4 - x^2) = 0$

$\therefore \lim_{x \to 2} f(x) = 0 = f(2)$

$\therefore$ continuous. ※

14. $g(x) = \begin{cases} \frac{1}{x+1}, & x \neq -1 \\ 0, & x = -1 \end{cases}$, $x = -1$

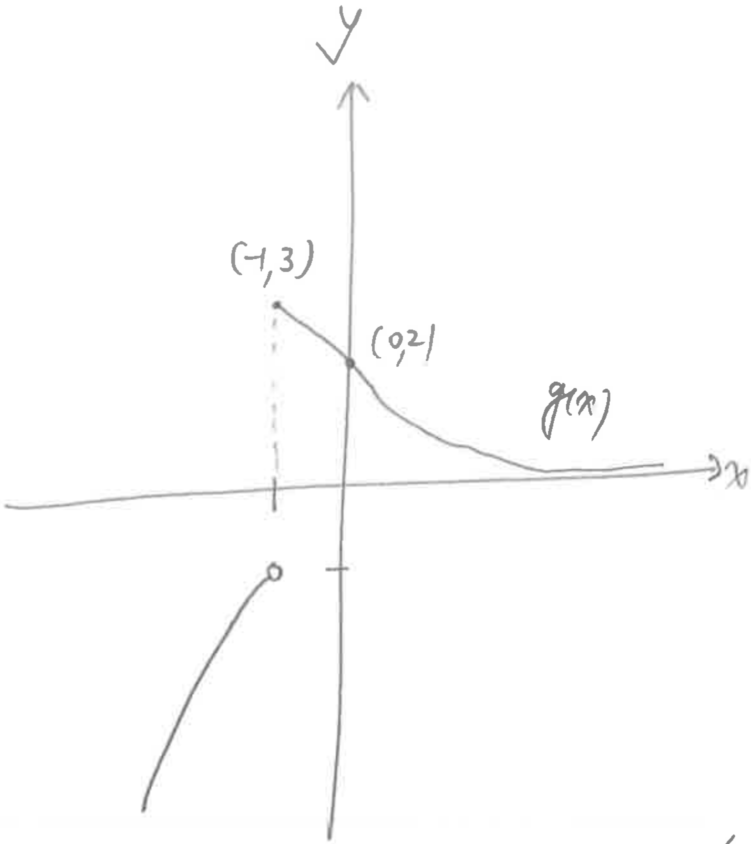
$\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} \frac{1}{x+1} = +\infty$, $\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} \frac{1}{x+1} = -\infty$, $g(-1) = 0$.

$\therefore$ infinite discontinuity. ✓

※

26.

$$g(x) = \begin{cases} -x^2, & x<-1 \\ 3, & x=-1 \\ 2-x, & -1<x\leq 1 \\ \frac{1}{x^2}, & 1<x \end{cases}$$



∴ jump discontinuity at $x=-1$

29. ① $dom(f)=[-3,3]$

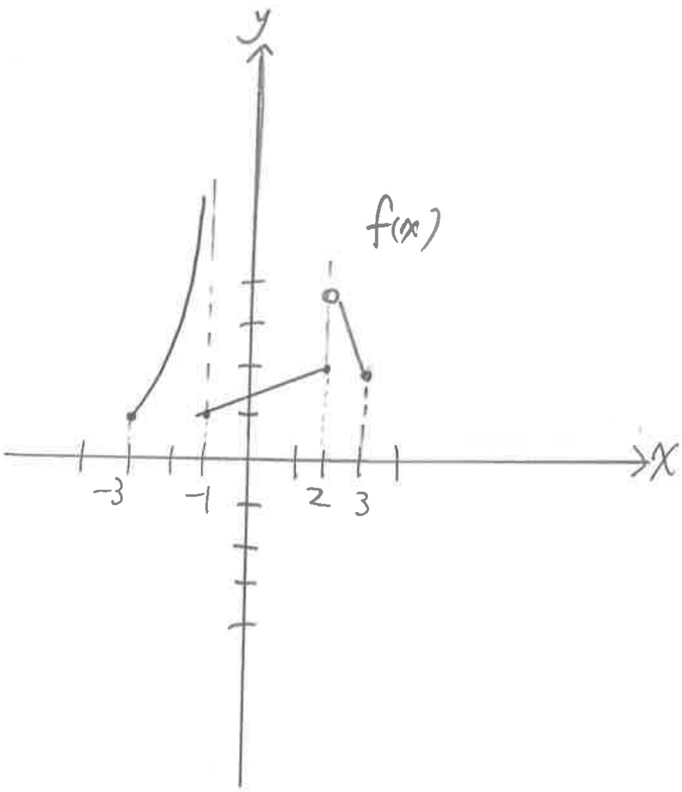
②

$f(-3)=f(-1)=1$

$f(2)=f(3)=2$

③ f has an infinite discontinuity at -1

and a jump discontinuity at 2



④

f is right continuous at -1

and left continuous at 2.

34

$$f(x)=\frac{(x-1)^2}{|x-1|}$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\frac{(x-1)^2}{x-1}=\lim_{x\to 1^+}(x-1)=0$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}\frac{(x-1)^2}{1-x}=\lim_{x\to 1^-}-(x-1)=0$$

Let $f(1)=0$. Then $\lim_{x\to 1} f(x)=0=f(1)$. So f is continuous at 1.

※

8. Salas 2.6 : 5, 9, 10, 15, 25, 26, 28.

5. $x^2 - 2 + \frac{1}{2x} = 0$, $\frac{1}{4} \le x \le 1$

<pf> Let $f(x) = x^2 - 2 + \frac{1}{2x}$. Then f is continuous on $[\frac{1}{4}, 1]$.

$f(\frac{1}{4}) = \frac{1}{16} > 0$, $f(1) = \frac{-1}{2} < 0$

Since $f(\frac{1}{4}) \times f(1) < 0$, by I.V.T., then $\exists\, c \in [\frac{1}{4}, 1]$, such that $f(c) = 0$.

∎

9. $f(x) = x^5 - 2x^2 + 5x$, $\exists\, c \in \mathbb{R}$, such that $f(c) = 1$.

<pf>

Since $f(0) = 0 < 1$, $f(1) = 4 > 1$, f is continuous on $[0, 1]$,

by I.V.T., then $\exists\, c \in (0, 1)$ such that $f(c) = 1$.

∎

8、Salas 2.6：5、9、10、15、25、26、28.

10、 $f(x) = \dfrac{1}{x-1} + \dfrac{1}{x-4}$, $\exists\, c \in (1,4)$ such that $f(c)=0$

<pf>

Since $f(2) = \frac{1}{2} > 0$, $f(3) = \frac{-1}{2} < 0$, f is continuous on $[2,3]$,

by I.V.T., then $\exists\, c \in (2,3) \subset (1,4)$ such that $f(c)=0$

▨

15、

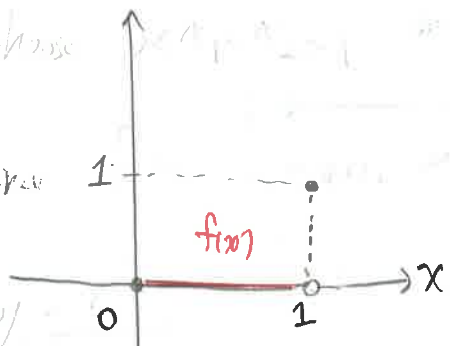
f is continuous on $(0,1)$, takes on the values 0 and 1,

↳ 在 (0,1) 只能取值 0 或 1 ※
(因為連續)

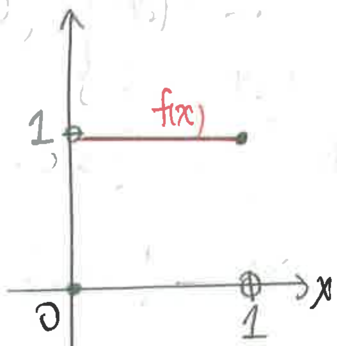
but does not take on the value $\frac{1}{2}$.

<sol>

Since f is defined on $[0,1]$, then



$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$



$f(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$

25、

If f is continuous on $[0,1]$ and $0 \leq f(x) \leq 1 \; \forall x \in [0,1]$,

then there exist at least one point c in $[0,1]$ at which $f(c) = c$.

<pf>

Let $g(x) = x - f(x)$ be defined on $[0,1]$.

case 1: If $f(0) = 0$ or $f(1) = 1$, then we done.

case 2: If $f(0) > 0$ and $f(1) < 1$, then we have

$g(0) = 0 - f(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$.

Since $g(0) < 0$, $g(1) > 0$, g is continuous on $[0,1]$, by I.V.T,

then $\exists \; c \in (0,1) \subset [0,1]$ such that $g(c) = 0$

$\Rightarrow c - f(c) = 0$

$\Rightarrow f(c) = c$.

∎

16.

If $f(x)$ and $g(x)$ are continuous on $[a,b]$, that $f(a) < g(a)$, $g(b) < f(b)$, then exists at least one number $c \in (a,b)$ such that $f(c) = g(c)$.

<pf>

Let $h(x) = f(x) - g(x)$ be defined on $[a,b]$.

Since f and g are continuous on $[a,b]$,

then $h = f - g$ is also continuous on $[a,b]$.

Since $h(a) = f(a) - g(a) < 0$, $h(b) = f(b) - g(b) > 0$, h is continuous on $[a,b]$,

by I.V.T., then $\exists\, c \in (a,b)$, such that $h(c) = 0$

$\Rightarrow f(c) - g(c) = 0$

$\Rightarrow f(c) = g(c)$ ∎

4.

$\forall a \in \mathbb{R}$, $\exists c \in \mathbb{R}$, such that $c^3 = a$.

<pf>

Let $f(x) = x^3$ be defined on $\mathbb{R}$.

case 1: $a=0$. Then we choose $c=0$ such that $f(0) = 0^3 = 0$.

case 2: $a>0$.

Since $f(0) = 0 < a$ and $f(a+1) = (a+1)^3 > a$, f is continuous on $\mathbb{R}$,

by I.V.T., then $\exists c \in (0, a+1)$ such that $f(c) = a$

$\Rightarrow c^3 = a$.

case 3: $a<0$.

Since $f(0) = 0 > a$ and $f(a-1) = (a-1)^3 < a-1 < a$ ($\because a-1 < -1$), f is continuous on $\mathbb{R}$,

by I.V.T., then $\exists c \in (a-1, 0)$ such that $f(c) = a$

$\Rightarrow c^3 = a$. ∎