國立成功大學水利及海洋工程學系
傑出系友頒獎典禮暨系友會會員大會

1. Prove that $f: A \Rightarrow B$ is continuous if and only if $f^{-1}((a,b))$ is open in A for every open interval $(a,b) \subset B$

pf: ($\Rightarrow$) Assume $f: A \Rightarrow B$ is continuous and that $(a,b) \subset B$ is open interval

Given a point $\alpha \in f^{-1}(a,b) \Rightarrow f(\alpha) \in (a,b)$

$\because$ $(a,b)$ is open $\Rightarrow \exists \varepsilon > 0$ s.t. $D(f(\alpha), \varepsilon) \subset (a,b)$

Since f is continuous at $\alpha \Rightarrow \exists \delta > 0$ s.t. $|x-\alpha| < \delta$, $|f(x)-f(\alpha)| < \varepsilon$

$\Rightarrow x \in D(\alpha, \delta)$, $f(x) \in D(f(\alpha), \varepsilon)$

In particular $f(x) \in (a,b)$ Thus $D(\alpha, \delta) \subset f^{-1}((a,b)) \Rightarrow f^{-1}((a,b))$ is open in A

($\Leftarrow$) Assume $f^{-1}((a,b))$ is open in A for every open interval $(a,b) \subset B$

$\forall \alpha \in A$ and $\varepsilon > 0$, $(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon)$ is open in B

As $\alpha \in f^{-1}(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon)$ is open in A

$\Rightarrow \exists\, \delta_\varepsilon > 0$ s.t. $(\alpha-\delta_\varepsilon, \alpha+\delta_\varepsilon) \subset f^{-1}(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon)$

$\Rightarrow x \in (\alpha-\varepsilon, \alpha+\varepsilon) \Rightarrow x \subset f^{-1}(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon) \Rightarrow f(x) \in f^{-1}(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon)$

$\Updownarrow$

$|x-\alpha| < \delta_\varepsilon \Rightarrow |f(x)-f(\alpha)| < \varepsilon$

QED

1. $(\Leftarrow)$ (修正)

$$x \in (\alpha - \overset{\delta_\varepsilon}{\cancel{\varepsilon}}, \alpha + \overset{\delta_\varepsilon}{\cancel{\varepsilon}}) \Rightarrow x \underset{\in}{\cancel{\subset}} f^{-1}(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon) \Rightarrow f(x) \in \cancel{f^{-1}}(f(\alpha)-\varepsilon, f(\alpha)+\varepsilon)$$

)

Let $m \in \mathbb{Q}$. $m = \frac{p}{q}$. which $p, q \in \mathbb{Z}$, $q > 0$

(claim $f(p) = p$ $\forall p \in \mathbb{Z}$)

case 1. : $p > 0$

when $p = 1$, $f(1) = 1$ holds

Suppose $p = k$, $f(k) = k$ holds

When $p = k+1$, $f(k+1) = f(k) + f(1) = k+1$

by induction, $f(p) = p$ $\forall p \in \mathbb{Z}^+$

case 2. : $p = 0$

$f(1) = f(0+1) = f(0) + f(1) = f(0) + 1 = 1$

$f(0) = 1 - 1 = 0$

case 3. : $p < 0$

$f(0) = f(-1+1) = f(-1) + f(1) = f(-1) + 1 = 0$

$f(-1) = -1$

when $-p = 1$, $f(-1) = -1$ holds.

Suppose $-p = k$, $f(-k) = -k$ holds

when $-p = k+1$, $f(-k-1) = f(-k) + f(-1) = -k-1$

by induction, $f(p) = p$ $\forall p \in \mathbb{Z}^-$

$\therefore f(p) = p$ $\forall p \in \mathbb{Z}$

$f\left(\frac{p}{q}\right) = f\left(\frac{1}{q}\right) + f\left(\frac{1}{q} + \cdots + \frac{1}{q}\right) = \cdots = f\left(\frac{1}{q}\right) + \cdots + f\left(\frac{1}{q}\right)$

claim $f(p) = q f\left(\frac{p}{q}\right)$ $\forall p \in \mathbb{Z}$, $q \in \mathbb{N}$

when $q = 1$, $f(p) = 1\, f\left(\frac{p}{1}\right)$ holds

Suppose $q = k$. $f(p) = k\, f\left(\frac{p}{k}\right)$ holds

when $q = k+1$. $f(p) = f\left(\frac{k}{k+1}p + \frac{1}{k+1}p\right)$

$= f\left(\frac{k}{k+1}p\right) + f\left(\frac{1}{k+1}p\right)$

$= k\, f\left(\frac{1}{k+1}p\right) + f\left(\frac{1}{k+1}p\right)$

$= (k+1) f\left(\frac{p}{k+1}\right)$.

by induction, $f(p) = q f\left(\frac{p}{q}\right)$ $\forall p \in \mathbb{Z}$, $q \in \mathbb{N}$.

$f(m) = f\left(\frac{p}{q}\right) = \frac{1}{q} f(p) = \frac{p}{q}$

(b) By definition: $f$ is continuous: $\lim_{y \to x} f(y)$ exists and $\lim_{y \to x} f(y) = f(x)$ $\forall x, y \in \mathbb{R}$

$\because$ By (a), we showed $f(x) = x$ for all rational numbers

$\therefore$ We can always find a sequence which approach any number in $\mathbb{R}$, including irrational numbers

$\therefore \exists y_n \in \mathbb{Q}$ and $y_n \to x \Rightarrow \lim_{y_n \to x} f(y_n) = \lim_{y_n \to x} y_n = x$

$\Rightarrow \lim_{y \to x} f(y) = x = f(x)$ $\forall x, y \in \mathbb{R}$

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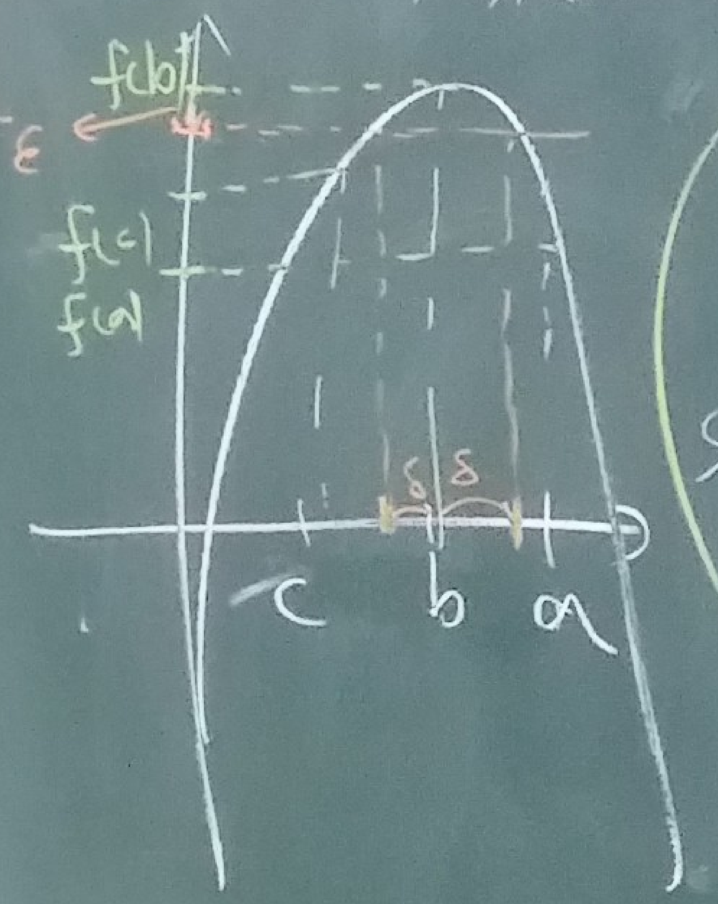
3、Prove that every continuous open mapping of $\mathbb{R}^1$ into $\mathbb{R}^1$ is monotonic.

$\varepsilon = \frac{1}{2}\min\left\{\begin{matrix} f(b)-f(a) \\ f(b)-f(c) \end{matrix}\right\}$

Assume that f isn't monotonic then there exists $x_0$ and $\delta > 0$ s.t



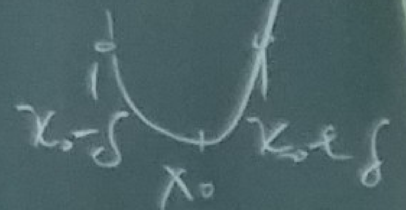

$\exists\ a > b > c$

s.t. $f(b) > f(c)$, $f(b) > f(a)$

OK!!

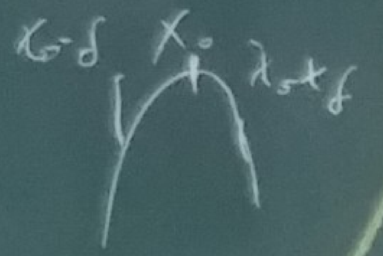
① $f'(x) < 0 \quad \forall x \in (x_0-\delta, x_0)$

$f'(x) > 0 \quad \forall x \in (x_0, x_0+\delta)$

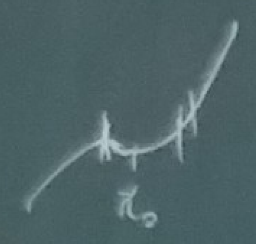



② $f'(x) > 0 \quad \forall x \in (x_0-\delta, x_0)$

$f'(x) < 0 \quad \forall x \in (x_0, x_0+\delta)$




∃不能使用!!




Case 1. $\exists\ \delta > 0$

$\forall x \in (x_0-\delta, x_0+\delta)$ (open)

s.t $f(x) \in [f(x_0), \mathrm{Max}\{f(x_0-\delta), f(x_0+\delta)\})$ (~open)

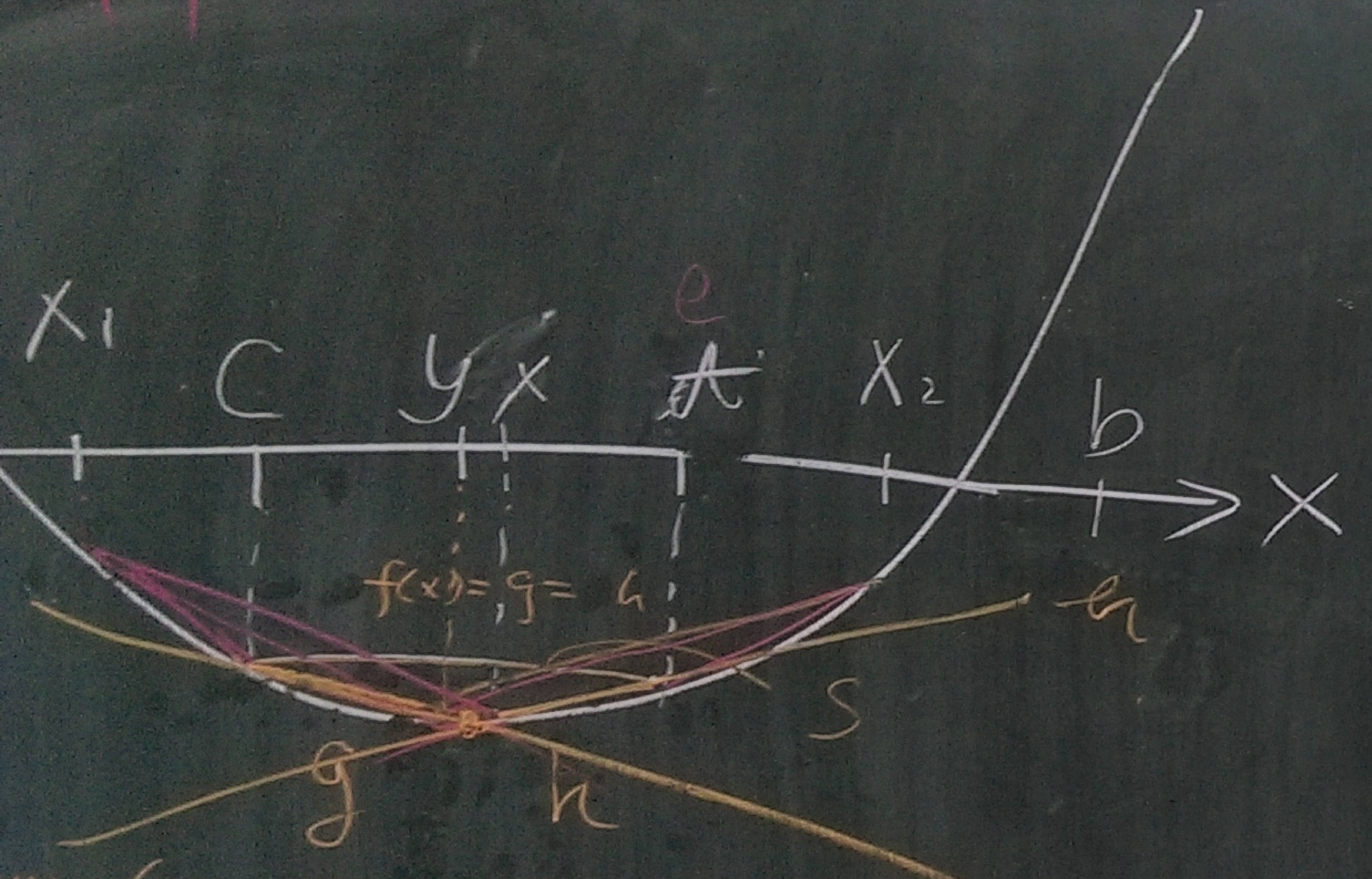
Case 2 $\exists\ \delta > 0$

$\forall x \in (x_0-\delta, x_0+\delta)$ (open)

$f(x) \in (\min\{f(x_0-\delta), f(x_0+\delta)\}, f(x_0)]$ (~open)

Hence, f(x) is not an open mapping of $\mathbb{R}^1$ into $\mathbb{R}^1$.

4.




$y \in (c, x)$ $\quad$ $y \in (x, t)$

$g > f > h$ $\quad$ $h > f > g$

$g(y) - g(x) = 0$

$f$ is defined on $(a, b)$

$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y)$

want to show $\lim_{y \to x} f(y) = f(x)$

$\forall \varepsilon > 0, \exists \delta$ s.t. $0 < |x - y| < \delta, |f(x) - f(y)| < \varepsilon$

given $x \in (a, b), \exists c, t \in (a, b)$ s.t. $c < x < t$

We have $x = \frac{x-c}{t-c} t + \left(1 - \frac{x-c}{t-c}\right) c$, $0 < \frac{x-c}{t-c} < 1$

So, $f(x) = \left(\frac{x-c}{t-c}\right) f(t) + \left(1 - \frac{x-c}{t-c}\right) f(c)$

Write the function

$$\frac{f(x) - f(c)}{x - c} \leq \frac{f(t) - f(c)}{t - c} \text{ or } \frac{f(t) - f(c)}{t - c} \leq \frac{f(t) - f(x)}{t - x}$$

e 換 t

Therefore $\forall x, y \in [c, t]$ let $x > y$, since $(a, b)$ open $\exists x_1, x_2 \in (a, b)$ s.t. $x_1 < c < t < x_2$, we get

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x_2) - f(y)}{x_2 - y} \leq \frac{f(x_2) - f(t)}{x_2 - t}$$

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(x) - f(x_1)}{x - x_1} \geq \frac{f(c) - f(x_1)}{c - x_1}$$

That is $\frac{|f(x) - f(y)|}{|x - y|} \leq M = \max\left\{\frac{|f(x_2) - f(t)|}{x_2 - t}, \frac{|f(c) - f(x_1)|}{c - x_1}\right\}$

$|f(x) - f(y)| \leq M|x - y| \leq M\delta$, let $\delta = \min\left\{\frac{\varepsilon}{M}, \frac{t - c}{2}\right\} > 0$

$\Rightarrow |f(x) - f(y)| \leq M \cdot \frac{\varepsilon}{M} = \varepsilon$

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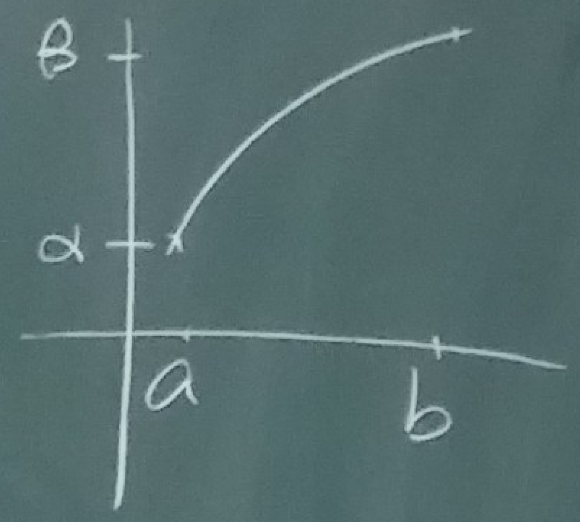
5、Prove $f(x)$ is monotonic in $[a,b]$ and satisfies the conclusion of intermediate value theorem, then $f(x)$ is continuous. For simplicity, you may assume the function is strictly monotonic




① assume $f(x)$ is strictly increasing

if $a<b$ let $f(a)=\alpha$ $f(b)=\beta$

then $\alpha<\beta$

(i) by IVT,

$\forall \gamma \in [\alpha,\beta]$,

that is $\alpha \le \gamma \le \beta$

$\exists c \in [a,b]$

s.t. $f(c)=\gamma$

so "surjection"

(ii)

$\forall p,q \in [a,b]$ and $p<q$

$\Rightarrow f(p)<f(q)$

so "one to one"

(i) and (ii)

$\therefore f$ is bijection on $[a,b]$

$f$ is invertible

$\Rightarrow f^{-1}([\alpha,\beta])=[a,b]$

by Thm (?)

$f: A \to B$ is continuous

$\Leftrightarrow f^{-1}((a,b))$ is open in $A$

$\Leftrightarrow f^{-1}([a,b])$ is closed in $A$

$\therefore f$ is continuous on $[a,b]$

② assume $f(x)$ is strictly decreasing

Similarly

6. $\forall y$ Lan.

$$f' = 0 \Rightarrow \lim_{x \to y} \frac{f(x) - f(y)}{x - y} = 0$$

$>0$.

$>0$ s.t $0 < |x-y| < \delta$, $\left| \frac{f(x) - f(y)}{x-y} \right| < \varepsilon$

$\longrightarrow |f(x) - f(y)| < |x-y| \varepsilon$

Also, $0 \leq |f(x) - f(y)| < |x-y| \varepsilon$

$\lim_{x \to y} 0 = \lim_{x \to y} |x-y| \varepsilon = 0$, by squeeze thm $\lim_{x \to y} |f(x) - f(y)| = 0 \Leftrightarrow \lim_{x \to y} f(x) = \boxed{\lim_{x \to y} f(y)} \Rightarrow$ const.

const.

$\lim_{x \to y} f(x) - f(y) = 0$

$\lim_{x \to y} f(x) - \lim_{x \to y} f(y) = 0$

Consider $\left| \frac{f(x) - f(y)}{x - y} \right| \leq |x-y|$

$0 \leq \left| \frac{f(x) - f(y)}{x-y} \right| \leq |x-y|$

since $\lim_{x \to y} 0 = 0$, $\lim_{x \to y} |x-y| = 0$

By squeeze thm

$\rightarrow \lim_{x \to y} \left| \frac{f(x) - f(y)}{x-y} \right| = 0$

$\Updownarrow$

$\lim_{x \to y} \frac{f(x) - f(y)}{x-y} = 0$, $\Rightarrow$ f is const

$f' = 0$

$\square$