Hw 8

1. Prove that $f: A \to B$ is continuous if and only if $f^{-1}((a,b))$ is open in A, for every open interval $(a,b) \subset B$.

<pf>

We know that $f: A \to B$ is continuous if and only if $f^{-1}(C)$ is open in A, for any open set $C \subset B$.

Claim : $f^{-1}(C)$ is open in A for any open set $C \subset B$ if and only if $f^{-1}((a,b))$ is open in A for any open interval $(a,b) \subset B$.

① Assume $f^{-1}(C)$ is open in A for any open set $C \subset B$.

For any open interval $(a,b) \subset B$. Then (a,b) is open set.

By our assumption, then we have $f^{-1}((a,b))$ is open in A.

② Assume $f^{-1}((a,b))$ is open in A for any open interval $(a,b) \subset B$.

For any open set $C \subset B$. Then $C = \bigcup_{k=1}^{\infty} (a_k, b_k)$, where (a_k, b_k) is open interval for all k.

$$f^{-1}(C) = f^{-1}\left(\bigcup_{k=1}^{\infty} (a_k, b_k)\right) = \bigcup_{k=1}^{\infty} f^{-1}((a_k, b_k))$$

By our assumption, then $f^{-1}((a_k, b_k))$ is open in A for all k.

So $f^{-1}(C) = \bigcup_{k=1}^{\infty} f^{-1}((a_k, b_k))$ is also open in A.

By ①, ②, then we complete this proof. ■

2. If $f: \mathbb{R} \to \mathbb{R}$ satisfies the equation $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and $f(1) = 1$.

(a) Find the values of $f(x)$ for all $x \in \mathbb{Q}$

(b) If in addition f is continuous, show that $f(x) = x$.

<sol>

(a)

① $f(n) = f(\underbrace{1+1+\cdots+1}_{(n\text{ times})}) = \underbrace{f(1) + f(1) + \cdots + f(1)}_{(n\text{ times})} = 1+1+\cdots+1 = n, \ \forall n \in \mathbb{N}.$

② $f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$

③ $f(0) = f(1+(-1)) = f(1) + f(-1) \Rightarrow 0 = 1 + f(-1) \Rightarrow f(-1) = -1$

④ $f(-n) = f(\underbrace{(-1)+(-1)+\cdots+(-1)}_{(n\text{ times})}) = \underbrace{f(-1) + f(-1) + \cdots + f(-1)}_{(n\text{ times})} = (-1)+(-1)+\cdots+(-1) = -n, \ \forall n \in \mathbb{N}.$

⑤ $\forall m \in \mathbb{Z}, n \in \mathbb{N} \setminus \{0\}$, then

$$m = f(m) = f(\underbrace{\tfrac{m}{n} + \tfrac{m}{n} + \cdots + \tfrac{m}{n}}_{(n\text{ times})}) = f(\tfrac{m}{n}) + f(\tfrac{m}{n}) + \cdots + f(\tfrac{m}{n}) = n \cdot f(\tfrac{m}{n})$$

$\Rightarrow f(\frac{m}{n}) = \frac{m}{n}$

By ①, ②, ③, ④, ⑤, then $f(x) = x, \ \forall x \in \mathbb{Q}$. ※

(b)

<pf> $\forall x \in \mathbb{Q}^c$, then $\exists \{r_n\} \subset \mathbb{Q}$ such that $\lim_{n\to\infty} r_n = x$.

Then $f(x) = f(\lim_{n\to\infty} r_n) \underset{(\because f \text{ is continuous})}{=\!=} \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} r_n = x$. So $f(x) = x, \ \forall x \in \mathbb{R}$. ▒

3. Rudin Ch4, #15 (See Def 4.28 for monotonic functions)

Call a mapping of X into Y open if $f(V)$ is an open set in Y whenever V is an open set in X.

Prove that every continuous open mapping of $\mathbb{R}^1$ into $\mathbb{R}^1$ is monotonic.

<pf>

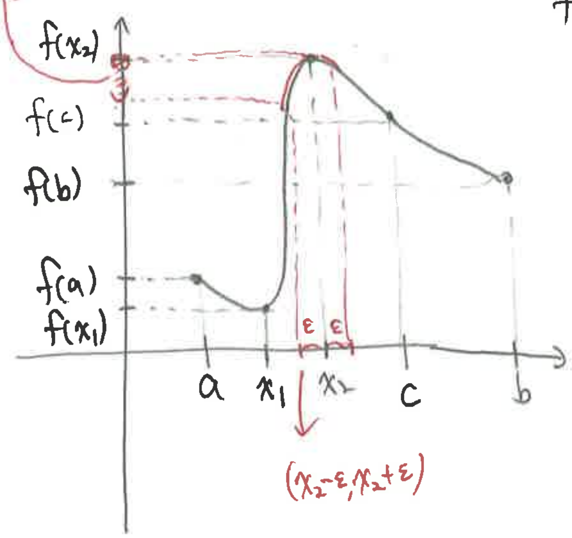
Suppose f is <u>not</u> monotonic, that is, $\exists\, a<c<b$, $a,b,c\in X$ such that $f(a)<f(c)$ and $f(b)<f(c)$.

Since $[a,b]$ is compact set and f is continuous on $[a,b]$, then f has maximum and minimum, that is, $\exists\, x_1, x_2 \in [a,b]$ such that $f(x_1) \le f(x) \le f(x_2)$ for any $x\in[a,b]$.

We choose $\varepsilon = \frac{1}{2}\min\{|x_1-x_2|,\ c-x_2\} > 0$, then $(x_2-\varepsilon, x_2+\varepsilon)$ is open interval.

Since $(x_2-\varepsilon, x_2+\varepsilon)$ is open interval and f is continuous on $(x_2-\varepsilon, x_2+\varepsilon)$, then $f((x_2-\varepsilon, x_2+\varepsilon))$ is interval and $f((x_2-\varepsilon, x_2+\varepsilon)) = (l, f(x_2)]$, or $f((x_2-\varepsilon, x_2+\varepsilon)) = [l, f(x_2)]$, where $l = \inf\limits_{x\in(x_2-\varepsilon,x_2+\varepsilon)} f(x)$.

$f((x_2-\varepsilon, x_2+\varepsilon))$



By our assumption, then $f((x_2-\varepsilon, x_2+\varepsilon))$ must be open, this is a contradiction.

Thus, f is monotonic.

4. Rudin Ch4, ※23 (Just prove the first statement)

A real-valued function f defined in (a,b) is said to be convex, if

$f(\lambda x+(1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$ whenever $a<x<b$, $a<y<b$, $0<\lambda<1$.

Prove that every convex function is continuous.

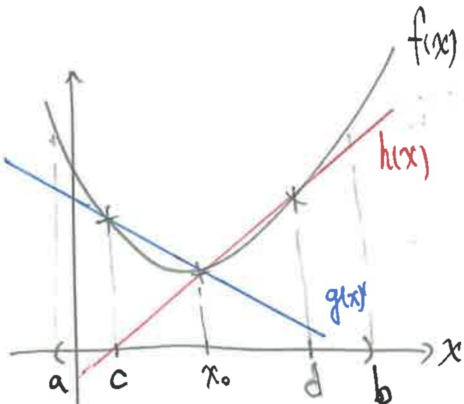
<pf> Let $a<c<x_0<d<b$.

Let $g(x)$ be the chord equation through $(c, f(c))$ and $(x_0, f(x_0))$ and $h(x)$ be the chord equation through $(x_0, f(x_0))$ and $(d, f(d))$.

Since f is convex, then $g(x) \ge f(x)$, $x\in[c,x_0]$ and $f(x) \le h(x)$, $x\in[x_0,d]$.

Now, claim: ① $f(x) \ge h(x)$, $x\in[c,x_0]$

② $g(x) \le f(x)$, $x\in[x_0,d]$



Let $h(x) = \dfrac{f(d)-f(x_0)}{d-x_0}(x-x_0)+f(x_0)$ and

$g(x) = \dfrac{f(x_0)-f(c)}{x_0-c}(x-x_0)+f(x_0)$. Then $h(x_0)=g(x_0)=f(x_0)$.

① $f(x)-h(x) = f(x) - \dfrac{f(d)-f(x_0)}{d-x_0}(x-x_0) - f(x_0) = \left[\dfrac{f(x)-f(x_0)}{x-x_0} - \dfrac{f(d)-f(x_0)}{d-x_0}\right](x-x_0)$

As $x\in[c,x_0)$, then $\dfrac{f(x)-f(x_0)}{x-x_0} \le \dfrac{f(d)-f(x_0)}{d-x_0}$, ($\because$ the slope of the chord is increasing on (a,b))

($\because c \le x < x_0 < d$)

$\Rightarrow f(x)-h(x) = \left[\dfrac{f(x)-f(x_0)}{x-x_0} - \dfrac{f(d)-f(x_0)}{d-x_0}\right](x-x_0) \ge 0$

So $f(x)-h(x) \ge 0$, as $x\in[c,x_0]$.

② $f(x) - g(x) = f(x) - \dfrac{f(x_0) - f(c)}{x_0 - c}(x - x_0) - f(x_0)$

$= \left[\dfrac{f(x) - f(x_0)}{x - x_0} - \dfrac{f(x_0) - f(c)}{x_0 - c}\right](x - x_0)$

As $x \in (x_0, d]$, then $\dfrac{f(x) - f(x_0)}{x - x_0} \geq \dfrac{f(x_0) - f(c)}{x_0 - c}$. ($\because$ the slope of the chord is increasing on (a,b))

($\because c < x_0 < x \leq d$)

$\Rightarrow f(x) - g(x) = \left[\dfrac{f(x) - f(x_0)}{x - x_0} - \dfrac{f(x_0) - f(c)}{x_0 - c}\right](x - x_0) > 0$

So $f(x) - g(x) \geq 0$, as $x \in [x_0, d]$

By ①, ②, then we have $h(x) \leq f(x) \leq g(x)$, as $x \in [c, x_0]$

and $g(x) \leq f(x) \leq h(x)$, as $x \in [x_0, d]$

③ As $x \to x_0^+$, then we have $\lim\limits_{x \to x_0^+} g(x) \leq \lim\limits_{x \to x_0^+} f(x) \leq \lim\limits_{x \to x_0^+} h(x_0)$

$\Rightarrow \lim\limits_{x \to x_0^+} f(x) = g(x_0) = h(x_0) = f(x_0)$

④ As $x \to x_0^-$, then we have $\lim\limits_{x \to x_0^-} h(x) \leq \lim\limits_{x \to x_0^-} f(x) \leq \lim\limits_{x \to x_0^-} g(x)$

$\Rightarrow \lim\limits_{x \to x_0^-} f(x) = h(x_0) = g(x_0) = f(x_0)$

By ③, ④, then $\lim\limits_{x \to x_0^+} f(x) = f(x_0) = \lim\limits_{x \to x_0^-} f(x) \Rightarrow \lim\limits_{x \to x_0} f(x) = f(x_0)$.

Thus, f is continuous at x_0. $\Rightarrow f$ is continuous on (a,b). ■

5.

Prove that if $f(x)$ is (strictly) monotonic on $[a,b]$ and satisfies the conclusion of intermediate value theorem, then $f(x)$ is continuous.

<pf> Let f be strictly increasing on $[a,b]$.

Let (c,d) be open interval and c, d be between $f(a)$ and $f(b)$, and $c<d$.

By I.V.T, then $\exists\ c', d' \in [a,b]$ such that $f(c') = c$ and $f(d') = d$.

Since f is increasing and $c<d$, then $c'<d'$. Claim: $f^{-1}((c,d)) = (c', d')$.

① $\forall x \in (c', d') \Rightarrow c' < x < d'$. Since f is monotonic, then $f(c') < f(x) < f(d')$.

$\Rightarrow c = f(c') < f(x) < f(d') = d$, that is, $c < f(x) < d \Rightarrow x \in f^{-1}((c,d))$.

So $(c', d') \subseteq f^{-1}((c,d))$.

② $\forall x \in f^{-1}((c,d)) \Rightarrow f(x) \in (\overset{f(c')}{\overset{\shortparallel}{c}}, \overset{f(d')}{\overset{\shortparallel}{d}})$. By I.V.T, then $\exists\ \alpha \in (c', d')$ such that

$f(\alpha) = f(x)$.

If f is strictly increasing on $[a,b]$, then f is one-to-one.

Since $f(\alpha) = f(x)$, then $\alpha = x \in (c', d')$. So $f^{-1}((c,d)) \subset (c', d')$.

By ①, ②, then $f^{-1}((c,d)) = (c', d')$.

By Hw 8, ※1, then f is continuous.

6. Rudin Ch5. #1 (You may assume the fact that $f' \equiv 0 \Rightarrow f$ is constant.)

Let f be defined for all real x, and suppose that $|f(x)-f(y)| \leq (x-y)^2$

for all real x and y.

Prove that f is constant.

<pf>

$\forall x \in \mathbb{R}$, then $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$

Since $|f(x+h)-f(x)| \leq (x+h-x)^2 = h^2$

$\Rightarrow -h^2 \leq f(x+h)-f(x) \leq h^2$

① As $h>0$, then $-h \leq \dfrac{f(x+h)-f(x)}{h} \leq h \Rightarrow \lim\limits_{h \to 0^+} \dfrac{f(x+h)-f(x)}{h} = 0$

② As $h<0$, then $-h \geq \dfrac{f(x+h)-f(x)}{h} \geq h \Rightarrow \lim\limits_{h \to 0^-} \dfrac{f(x+h)-f(x)}{h} = 0$

By ①, ②, then $\lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h} = 0$

$\Rightarrow f'(x) = 0$, $\forall x \in \mathbb{R}$

$\Rightarrow f$ is constant. $\blacksquare$

7. Salas §3-1 ※ 9, 14, 18, 35, 45, 52, 59

9. $f(x) = \frac{1}{x^2}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{hx^2 (x+h)^2} = \lim_{h\to 0} \frac{-2x-h}{x^2 (x+h)^2} = \frac{-2x}{x^4} = \frac{-2}{x^3}$$

※

14. $f(x) = 5 - x^4, \quad c = -1$

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{5 - (-1+h)^4 - 4}{h}$$

$$= \lim_{h\to 0} \frac{5 - (h^4 + 4h^2 + 1 - 4h^3 - 4h + 2h^2) - 4}{h}$$

$$= \lim_{h\to 0} (-h^3 - 4h + 4h^2 + 4 - 2h)$$

$$= 4$$

※

18. $f(x)=\sqrt{x}$, $c=4$, find tangent line ?

<sol>

$$f'(x)=\frac{1}{2\sqrt{x}} \Rightarrow f'(4)=\frac{1}{4} \quad \text{and} \quad f(4)=2$$

$\therefore$ tangent line : $y=\frac{1}{4}(x-4)+2$

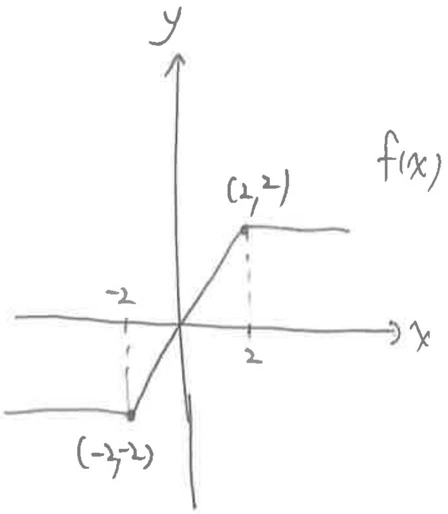
$$y=\frac{1}{4}x+1$$

$$4y=x+4$$

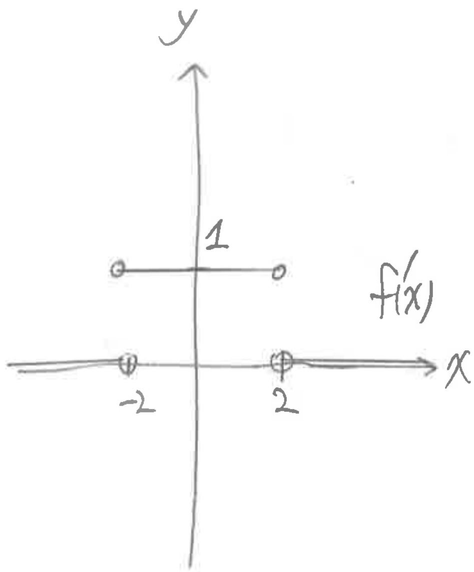
$$\underline{x-4y+4=0}$$

※

35.



$\Rightarrow$



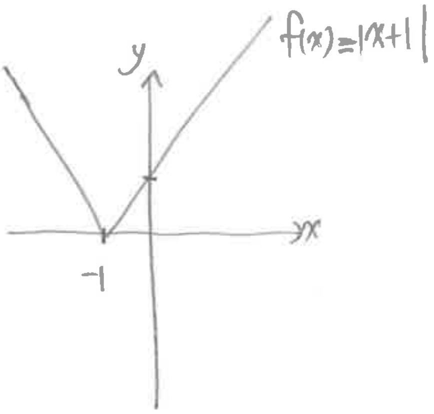
$$f'(x)=\begin{cases} 0, & x>2 \\ 1, & -2<x<2 \\ 0, & x<-2 \end{cases}$$

※

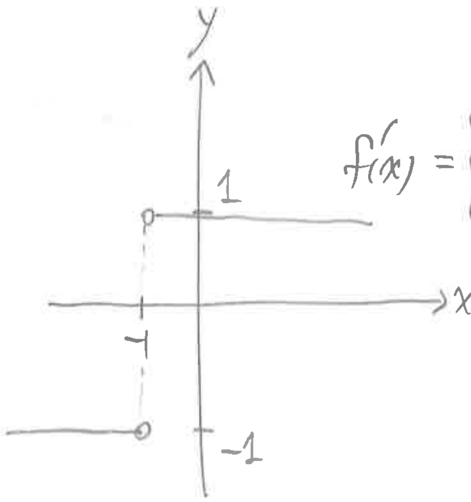
45.
$f'(x)$ exists for all $x \neq -1$; $f'(-1)$ does not exist.

<sol>

① $f(x) = |x+1|$

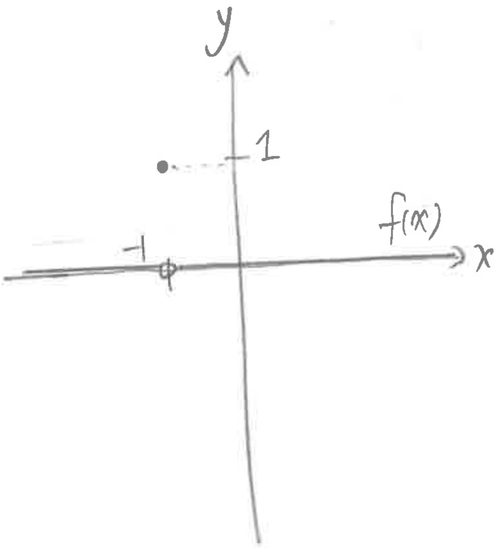


$\Rightarrow$

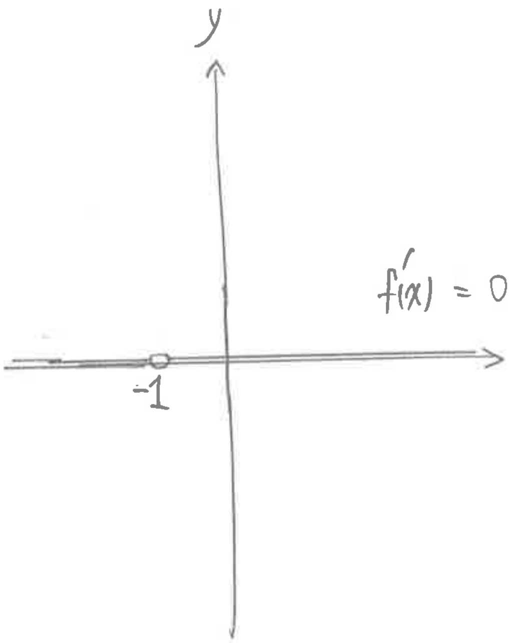


$$f'(x) = \begin{cases} 1, & x > -1 \\ -1, & x < -1 \end{cases}$$

② $$f(x) = \begin{cases} 0, & x \neq -1 \\ 1, & x = -1 \end{cases}$$



$\Rightarrow$



$f'(x) = 0$, as $x \neq -1$.

52.

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}, \quad g(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}$$

(a) Show that f is not differentiable at 0

(b) Show that g is differentiable at 0 and give $g'(0)$.

<pf>

(a)

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h} = \begin{cases} \lim\limits_{h \to 0} \frac{h}{h} = 1, & h \in \mathbb{Q} \\ \lim\limits_{h \to 0} \frac{0}{h} = 0, & h \in \mathbb{Q}^c \end{cases}$$

So, $f'(0)$ does not exist $\blacksquare$

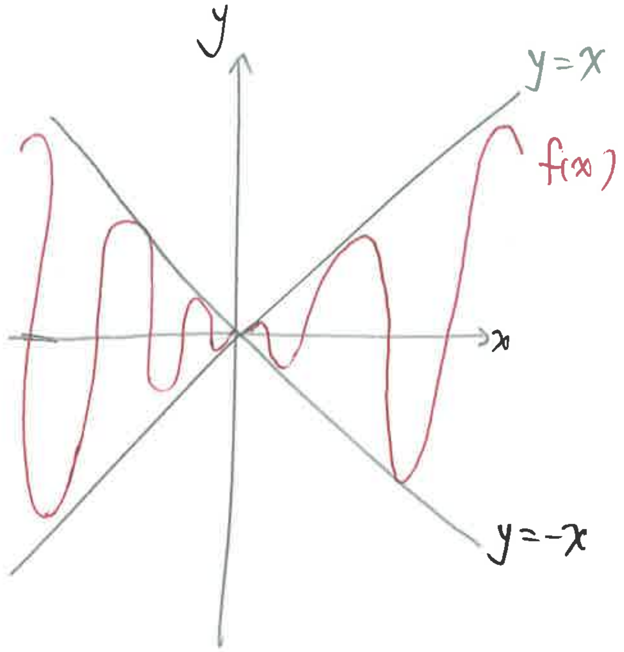
(b)

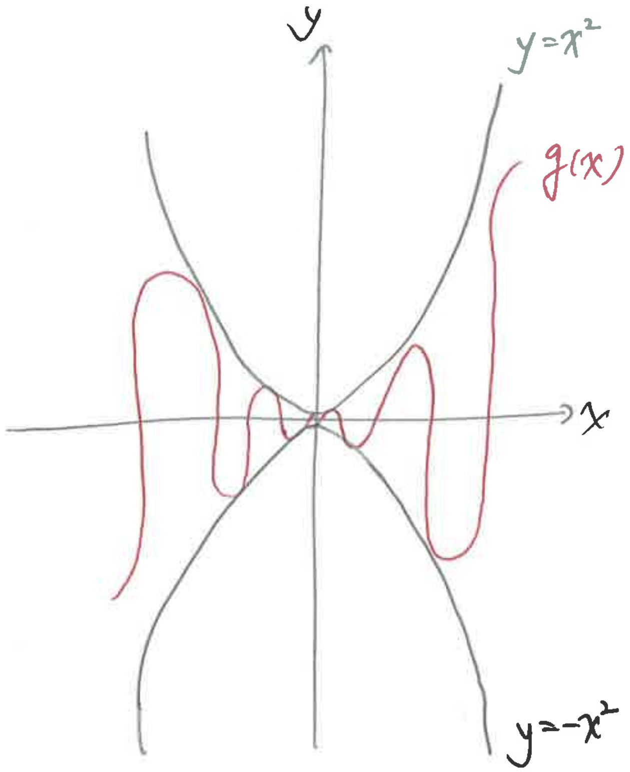
$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{g(h) - 0}{h} = \begin{cases} \lim\limits_{h \to 0} \frac{h^2}{h} = \lim\limits_{h \to 0} h = 0, & h \in \mathbb{Q} \\ \lim\limits_{h \to 0} \frac{0}{h} = 0, & h \in \mathbb{Q}^c \end{cases}$$

So, $g'(0) = 0$, that is, g is differentiable at 0. $\blacksquare$

59.

$$f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{and} \quad g(x) = x \cdot f(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$





(a) Show that f and g are both continuous at 0

(b) Show that f is <u>not</u> differentiable at 0

(c) Show that g is differentiable at 0 and give $g'(0)$.

<pf> (a) As $x \neq 0$, then $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$.

Then we have $0 \leq |f(x)| = \left|x \sin\left(\frac{1}{x}\right)\right| \leq |x|$, as $x \neq 0$

and $0 \leq |g(x)| = \left|x^2 \cdot \sin\left(\frac{1}{x}\right)\right| \leq |x|^2$, as $x \neq 0$.

So $\lim_{x \to 0} f(x) = 0 = f(0)$ and $\lim_{x \to 0} g(x) = 0 = g(0)$.

So f and g are continuous at 0. ■

(b)

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \frac{h\sin(\frac{1}{h})}{h}$$

$$= \lim_{h\to 0} \sin(\frac{1}{h}) \text{ does } \underline{\text{not}} \text{ exist.}$$

So f is <u>not</u> differentiable at 0. ▨

(c)

$$g'(0) = \lim_{h\to 0} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0} \frac{g(h)}{h} = \lim_{h\to 0} \frac{h^2\sin(\frac{1}{h})}{h}$$

$$= \lim_{h\to 0} h\cdot\sin(\frac{1}{h}) = f(0) = 0$$

So g is differentiable at 0 and $g'(0) = 0$. ▨