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1. $g$ is a real function on $\mathbb{R}^1$

$|g'| \le M \iff -M \le g' \le M$

$f(x) = x + \varepsilon g(x)$

$f'(x) = 1 + \varepsilon g'(x)$, $|g'(x)| \le M$

choose $0 < \varepsilon < \frac{1}{M} \iff |\varepsilon g'(x)| < 1$

$\Rightarrow 0 < 1 + \varepsilon g'(x) < 2$, $\forall x \in \mathbb{R}$

$\Rightarrow 0 < f'(x) < 2$, $\forall x \in \mathbb{R}$

That is, $\forall a, b \in \mathbb{R}$, $b > a$

$f(b) > f(a)$

$\Rightarrow f$ is one-to-one

QED.

f differentiable

and $f'(x) \xrightarrow{x\to+\infty} 0$

Put $g(x)=f(x+1)-f(x)$

Prove $g(x) \xrightarrow{x\to+\infty} 0$

Pf. Since $f(x) \xrightarrow{x\to+\infty} 0$ for every $x>0$

$\forall \varepsilon>0,\ \exists M>0,\ x \geq M$

s.t. $|f(x)-0|<\varepsilon$.

By MVT

$f:[x, x+1] \longrightarrow \mathbb{R}$, diff.

$\exists c \in (x, x+1)$

s.t. $\dfrac{f(x+1)-f(x)}{x+1-x} = f'(c)$

$\Rightarrow f(x+1)-f(x) = f'(c)$

$\Rightarrow g(x) = f'(c)$

$\forall \varepsilon>0,\ \exists M>0,\ x \geq M$

s.t. $|f(x+1)-f(x)| = |f'(c)| = |g(x)-0| < \varepsilon$

that is $g(x) \to 0$ as $x \to +\infty$

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$g(x) = \dfrac{f(x)}{x}$

$f(x)$ is continuous on $[0, x]$ and is diff on $(0, x)$, by MVT

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}$$

$\exists\ 0 < c < x$ and $f'$ monotonic increasing

$\therefore f'(c) < f'(x)$

$\because x > 0$, $f'(c) = \dfrac{f(x)}{x} < f'(x)$ (MVT)

$\therefore f(x) < x f'(x)$

$$\Rightarrow g'(x) = \frac{x f'(x) - f(x)}{x^2} > 0$$

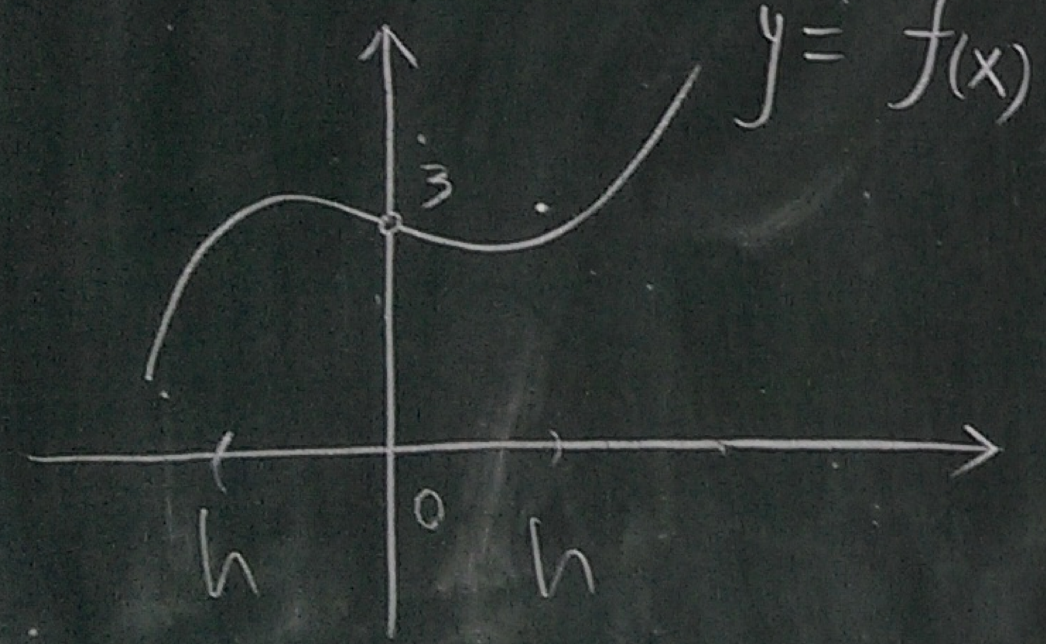
$\therefore g$ is monotonic increasing.

4. Let $f$ be continuous real function on $\mathbb{R}^1$

where it's known that $f'(x)$ exists for all

$x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$

Does it follow that $f'(0)$ exists? Yes




pf $\lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{(0+h) - 0} = \lim\limits_{h \to 0} \frac{f(h) - f(0)}{h}$ — ⊛

$\because$ $f$ is continous on $\mathbb{R}$ and

$f$ is cont on $[a,b]$ differentiable on $\mathbb{R} \setminus \{0\}$

diff on $(a,b)$ By Mean value thm

$\exists\, c \in (a,b)$

s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$

Case 1. Let $0 < h$, $\exists\, c_1 \in (0,h)$ s.t.

$f'(c_1) = \frac{f(h) - f(0)}{h - 0} \Rightarrow h f'(c_1) = f(h) - f(0)$

By ⊛ we get $\frac{h f'(c_1)}{h} = \lim\limits_{h \to 0} f'(c_1) = \lim\limits_{c_1 \to 0} f'(c_1)$

since $f'(x) \to 3$ as $x \to 0$

thus $\lim f'(c_1) = 3$

Case 2. Let $0 > h$, $\exists\, c_2 \in (h,0)$ s.t.

$f'(c_2) = \frac{f(0) - f(h)}{0 - h} \Rightarrow h f'(c_2) = -f(0) + f(h)$

By ⊛, we get $\frac{h f'(c_2)}{h} = \lim\limits_{h \to 0} f'(c_2)$

since $f'(x) \to 3$ as $x \to 0$ $= \lim\limits_{c_2 \to 0} f'(c_2)$

thus, $\lim\limits_{h \to 0} f'(c_2) = 3$

thus, $f'(0)$ exists #

5. (a)

<i> $\boxed{a>0}$ : $f(x)=x^{a}\sin x^{-c}$

$\because -1\le \sin y\le 1\ \forall y$ and $\lim_{x\to 0} x^{a}=0$

$\therefore \lim_{x\to 0}(x^{a}\sin x^{-c})=0=f(0)$ conti

<ii> $a=0$ : $f(x)=\sin x^{-c}$

$\lim_{x\to 0}(\sin x^{-c})=\lim_{x\to 0}(\sin \frac{1}{x^{c}})$ DNE not conti

<iii> $a<0$ $f(x)=x^{a}\sin x^{-c}$

$\lim_{x\to 0} x^{a}=\pm\infty$, $\lim_{x\to 0}\sin x^{-c}$ DNE

$\Rightarrow \lim_{x\to 0} x^{a}\sin x^{-c}$ DNE ~~$\ne 0$~~ not conti

$\therefore$ f conti iff $a>0$ #

(b) $f(x)=\begin{cases} x^{a}\sin x^{-c}, & x\ne 0 \\ 0, & x=0 \end{cases}$, $c>0$, $x\in[-1,1]$, $x\in(-1,1)$

$$f'(0)=\lim_{h\to 0}\frac{(0+h)^{a}\sin(0+h)^{-c}-\overset{0}{f(0)}}{h}$$

$$=\boxed{\lim_{h\to 0}\frac{h^{a}\sin h^{-c}}{h}}$$

$$=\lim_{h\to 0}(h^{a-1}\sin h^{-c})=0$$

Similary to (a.) $\lim_{h\to 0} h^{a-1}\sin h^{-c}$ exises iff $a-1>0 \Leftrightarrow a>1$

(c). $f'$ bdd $\Leftrightarrow a\ge 1+c$ and $f'(0)=0$

$f'(x)=ax^{a-1}\sin x^{-c}-cx^{a-c-1}\cos x^{-c}$, $\forall x\in(-1,1)\setminus\{0\}$

$=\sqrt{a^{2}x^{2a-2}+c^{2}x^{2a-2c-2}}\,\sin(x^{-c}-y)$ $\quad y=\sin^{-1}\left(\frac{-cx^{a-c-1}}{\sqrt{\ }}\right)=\cos^{-1}\left(\frac{ax^{a-1}}{\sqrt{\ }}\right)$

$\Rightarrow f'(x)\in\left[-\sqrt{a^{2}x^{2a-2}+c^{2}x^{2a-2c-2}},\ \sqrt{a^{2}x^{2a-2}+c^{2}x^{2a-2c-2}}\right]$

$\because \begin{cases} -1\le \sin x^{-c}, \cos x^{-c}\le 1 \\ ax^{a-1} \xrightarrow{a\ \text{fixed},\ x\to 0} \boxed{0} \end{cases}$ $\therefore$ we only need to discuss about $cx^{a-c-1}$

$, a-1=0, ax^{a-1} \to a$

(i) $a>1+c \Rightarrow a-c-1>0$, $cx^{a-c-1}\xrightarrow{x\to 0}0$ $\quad f'(x)\in[-\sqrt{\ },\sqrt{\ }]$

(ii) $a=1+c \Rightarrow a-c-1=0$, $cx^{a-c-1}=c\ \forall x$ $\quad f'(x)\in\left[-\sqrt{a^{2}x^{2a-2}+c},\ \sqrt{a^{2}x^{2a-2}+c}\right]$

(iii) $a<1+c \Rightarrow a-c-1<0$, $cx^{a-c-1}\xrightarrow{x\to 0}\pm\infty$ has no bound

$\therefore f'$ bdd iff $a\ge 1+c$ and $f'(x)\in[-\sqrt{a^{2}+c^{2}},\sqrt{a^{2}+c^{2}}]\ \forall x\in[-1,1]$

$\in[-a-c,\ a+c]$

(d) f

f'(x

want

That's to

discuss $x\to 0$

(i) a >

$\Rightarrow \{$ a

a

(c). $f'$ bdd $\Leftrightarrow a \geq 1+c$ and $f'(0)=0$

$[-1,1]$

$(-1,1)$

$f'(x) = ax^{a-1}\sin x^{-c} - cx^{a-c-1}\cos x^{-c}, \ \forall x \in (-1,1)\setminus\{0\}$

$= \sqrt{a^2x^{2a-2}+c^2x^{2a-2c-2}}\ \sin(x^{-c}-y)$ $\quad y=\sin^{-1}\left(\frac{-cx^{a-c-1}}{\sqrt{\ }}\right) = \cos^{-1}\left(\frac{ax^{a-1}}{\sqrt{\ }}\right)$

$\Rightarrow f'(x) \in \left[-\sqrt{a^2x^{2a-2}+c^2x^{2a-2c-2}}, \sqrt{a^2x^{2a-2}+c^2x^{2a-2c-2}}\right]$

$\because \begin{cases} -1 \leq \sin x^{-c}, \cos x^{-c} \leq 1 \\ ax^{a-1} \xrightarrow{a \text{ fixed}, x\to 0} 0 \\ a-1=0,\ ax^{a-1} \longrightarrow a \end{cases}$ $\therefore$ we only need to discuss about $cx^{a-c-1}$

(i) $a>1+c \Rightarrow a-c-1>0,\ cx^{a-c-1} \xrightarrow{x\to 0} 0$ $\quad f'(x) \in [-\sqrt{\ }, \sqrt{\ }]$

(ii) $a=1+c \Rightarrow a-c-1=0,\ cx^{a-c-1} = c \ \forall x$ $\quad f'(x) \in [-\sqrt{a^2x^{2a-2}+c}, \sqrt{a^2x^{2a-2}+c}]$

(iii) $a<1+c \Rightarrow a-c-1<0,\ cx^{a-c-1} \xrightarrow{x\to 0} \pm\infty$ has no bound

exises

$\therefore f'$ bdd iff $a \geq 1+c$ and $f'(x) \in [-\sqrt{a^2+c^2}, \sqrt{a^2+c^2}]\ \forall x \in [-1,1]$

$\in [-a-c, a+c]$

(d) $f'$ conti $\Leftrightarrow a>1+c$ $\quad \therefore f'$ conti iff $a>1+c$

$f'(x) = ax^{a-1}\sin x^{-c} - cx^{a-c-1}\cos x^{-c}$

want to show $\lim_{x\to k}\left(ax^{a-1}\sin x^{-c} - cx^{a-c-1}\cos x^{-c}\right)$

$= ak^{a-1}\sin k^{-c} - ck^{a-c-1}\cos k^{-c} \quad \forall k \in (-1,1)$

That's to show $\begin{cases} \lim_{x\to k}(ax^{a-1}\sin x^{-c}) = ak^{a-1}\sin k^{-c} & — * \\ \lim_{x\to k}(-cx^{a-c-1}\cos x^{-c}) = -ck^{a-c-1}\cos k^{-c} & — ** \end{cases}$

discuss $k=0$

(i) $a>1+c$ $\Rightarrow \begin{cases} a-1>c>0 \\ a-c-1>0 \end{cases} \Rightarrow \begin{cases} * \text{ holds} \\ ** \text{ holds} \end{cases}$ (by (a))

(ii) $a=1+c$ $\Rightarrow \begin{cases} a-1=c>0 \\ a-c-1=0 \end{cases} \Rightarrow \begin{cases} * \text{ holds} \\ * \text{ doesn't hold} \end{cases}$ (by (a))

(iii) $a<1+c$ $\Rightarrow \begin{cases} a-1<c \\ a-c-1<0 \end{cases} \Rightarrow \begin{cases} * \text{ need to be discussed} \\ ** \text{ doesn't hold} \end{cases}$ (by (a))

Prove (a) $\sin x \le x \quad \forall x \ge 0$

(b) $x \le \tan x \quad \forall x \in [0, \frac{\pi}{2}]$

(c) $x \ge \log x \quad \forall x > 0$

(a) Let $g(x) = x - \sin x \quad \forall x \ge 0$

Then $g(0) = 0$ and $g'(x) = 1 - \cos x \ge 0$

$\therefore$ $g$ is always nondecreasing on $[0, \infty)$

$\Rightarrow x - \sin x \ge 0$ and $x \ge \sin x$ for $x \ge 0$

(b) $\tan x \ge x$ for $x \in [0, \frac{\pi}{2}]$

Let $g(x) = \tan x - x \quad \forall x \in [0, \frac{\pi}{2})$

Then $g(0) = 0$ and $g'(x) = \sec^2 x - 1$

$= \left(\frac{1}{\cos^2 x}\right) - 1$, sine $\cos^2 x \le 1$, $\frac{1}{\cos^2 x} \ge 1$

$\ge 0 \quad x \in [0, \frac{\pi}{2})$

Thus $g$ is always nondecreasing on $[0, \frac{\pi}{2}]$

Thus $\tan x - x \ge 0$ and $\tan x \ge x$ for $x \in [0, \frac{\pi}{2}]$

(c)

$e^x > 2^x = (1+1)^x \ge 1 + x > x \quad \forall x > 0$

Thus $\log e^x > \log x \Rightarrow x > \log x \quad \forall x > 0$