Hw 9

1.

Rudin Ch5 #3.

Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$.). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M.)

<pf> Let $f(x) = f(y)$ for any $x, y \in \mathbb{R}$.

For any given $\varepsilon > 0$, then we have $x + \varepsilon g(x) = y + \varepsilon g(y)$

$$\Rightarrow \quad x - y = \varepsilon (g(y) - g(x)) \Rightarrow |x-y| = \varepsilon \, |g(y) - g(x)|.$$

By Mean-Value Theorem, then $|g(y) - g(x)| = |g'(t)| \cdot |x-y|$ for some t between x and y.

$$\Rightarrow |x-y| = \varepsilon \cdot |g(y) - g(x)| = \varepsilon \cdot |g'(t)| \cdot |y-x| \leq \varepsilon \cdot M \cdot |x-y|.$$

$$\Rightarrow |x-y| \cdot (1 - \varepsilon M) \leq 0$$

We choose $\varepsilon > 0$ such that $0 < \varepsilon M < \frac{1}{2} \Rightarrow \frac{1}{2} < 1 - \varepsilon M$.

Then $|x-y| = 0 \Rightarrow x = y$.

So f is one-to-one.

2. Rudin Ch5 #5

Suppose f is defined and differentiable for every $x>0$, and $f'(x) \to 0$ as $x \to +\infty$.

Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

<pf>

Given $\varepsilon > 0$, since $\lim_{x \to +\infty} f'(x) = 0$, then $\exists M > 0$ such that if $x > M$,

we have $|f'(x)| < \varepsilon$.

By Mean-Value Theorem, then $|g(x)| = |f(x+1) - f(x)| = |f'(t)| \cdot |x+1-x|$,

for some $x < t < x+1$, for any $x \in \mathbb{R}$

$\Rightarrow |g(x)| = |f'(t)|$ for some $x < t < x+1$, for any $x \in \mathbb{R}$.

If $x > M$, then we have $|g(x)| = |f'(t)|$ for some $M < x < t < x+1$.

$\Rightarrow |g(x)| = |f'(t)| < \varepsilon$.

So $\lim_{x \to +\infty} g(x) = 0$. ∎

3. Rudin Ch5 ※6

Suppose (a) f is continuous for $x \geq 0$, (b) $f'(x)$ exists for $x > 0$,

(c) $f(0) = 0$ (d) f' is monotonically increasing. Put $g(x) = \frac{f(x)}{x}$ $(x > 0)$ and

prove that g is monotonically increasing.

<pf> Let $0 < x < y$. Then $g(y) - g(x) = \frac{f(y)}{y} - \frac{f(x)}{x} = \frac{x f(y) - y f(x)}{xy}$

$$= \frac{x(f(y) - f(x)) + f(x)(x - y)}{x \cdot y}.$$

By M.V.T., then $f(y) - f(x) = f'(t) \cdot (y - x)$ for some $x < t < y$.

By M.V.T., then $f(x) - 0 = f(x) - f(0) = f'(s)(x - 0)$ for some $0 < s < x$.

Then we have $g(y) - g(x) = \frac{x \cdot f'(t)(y - x) + f'(s) \cdot x \cdot (x - y)}{xy}$

$$= \frac{x \cdot (y - x) \cdot (f'(t) - f'(s))}{x \cdot y}, \quad 0 < x < y.$$

Since f' is monotonically increasing and $0 < s < x < t < y$, then $f'(s) \leq f'(t)$.

Thus, we have $g(y) - g(x) \geq 0$, $0 < x < y$. $\Rightarrow$ $g(y) \geq g(x)$, $0 < x < y$.

So g is monotonically increasing.

4.

Rudin Ch5 #9

Let f be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$.

Does it follow that $f'(0)$ exist?

<pf> $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$

① As $h > 0$, since f is continuous on $[0, h]$ and is differentiable on $(0, h)$,

by M.V.T, then $f(h) - f(0) = f'(t) \cdot (h - 0)$ for some $0 < t < h$.

$$\Rightarrow \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{t \to 0^+} \frac{f'(t) \cdot h}{h} = \lim_{t \to 0^+} f'(t) = 3.$$

② As $h < 0$, since f is continuous on $[h, 0]$ and is differentiable on $(h, 0)$,

by M.V.T., then $f(h) - f(0) = f'(t) \cdot (h - 0)$ for some $h < t < 0$.

$$\Rightarrow \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{t \to 0^-} \frac{f'(t) \cdot h}{h} = \lim_{t \to 0^-} f'(t) = 3.$$

By ①, ②, then $\lim_{h \to 0^+} \dfrac{f(h) - f(0)}{h} = 3 = \lim_{h \to 0^-} \dfrac{f(h) - f(0)}{h}$

$$\Rightarrow f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 3. \text{ (exist!!)}$$

I.

Rudin Ch5 #13(a)-(d)

Suppose a and c are real numbers, $c>0$, and f is defined on $[-1,1]$ by

$$f(x)=\begin{cases} x^a \cdot \sin(x^{-c}), & \text{if } x\neq 0, \\ 0, & \text{if } x=0. \end{cases}$$

Prove the following statements:

(a) f is continuous if and only if $a>0$.

(b) $f'(0)$ exists if and only if $a>1$.

(c) f' is bounded if and only if $a\geq 1+c$.

(d) f' is continuous if and only if $a>1+c$.

<pf> (a)

① Assume $a>0$.

Claim: $f(x)$ is continuous at $x=0$.

As $x\neq 0$, we have $0\leq |x^a\cdot \sin(x^{-c})| \leq |x|^a$ and $0<|x|\leq 1$.

Since $a>0$, then $\lim_{x\to 0}|x|^a=0$.

Thus, $\lim_{x\to 0}|x^a\cdot\sin(x^{-c})|=0 \Rightarrow \lim_{x\to 0} f(x)=\lim_{x\to 0} x^a\cdot \sin(x^{-c})=0=f(0)$.

So $f(x)$ is continuous at $x=0$. —(*)

As $a>0$, then x^a is continuous for $x\neq 0$. So $f(x)$ is continuous for $x\neq 0$. (**)

By (*) (**), then f is continuous on $[-1,1]$.

② Assume f is continuous on $[-1,1] \Rightarrow f$ is continuous at $x=0$.

$$\lim_{x\to 0} f(x) = f(0) \Rightarrow \lim_{x\to 0} x^a \cdot \sin(x^{-c}) = 0.$$

(1) $a=0$, $\lim_{x\to 0} x^a \cdot \sin(x^{-c}) = \lim_{x\to 0} \sin(x^{-c})$ does not converge.

(2) $a<0$, $\lim_{x\to 0} x^a \cdot \sin(x^{-c})$ does not converge.

(3) $a>0$, let $x\neq 0$, $0 \leq |x^a \cdot \sin(x^{-c})| \leq |x|^a$, and $\lim_{x\to 0} |x|^a = 0$ ($\because 0 \leq |x| \leq 1$),

then $\lim_{x\to 0} |x^a \cdot \sin(x^{-c})| = 0 \Rightarrow \lim_{x\to 0} x^a \cdot \sin(x^{-c}) = 0$.

Thus, we have $a>0$.

By ①, ②, then f is continuous $\Leftrightarrow a>0$. ■

(b) ③ Assume $a>1 \Rightarrow a-1>0$.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^a \cdot \sin(h^{-c})}{h} = \lim_{h\to 0} h^{a-1} \cdot \sin(h^{-c}) \overset{\because ①}{=\!=} 0 \text{ exists.}$$

④ Assume $f'(0)$ exists.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} h^{a-1} \cdot \sin(h^{-c}) \text{ exists.}$$

(1) $a=1$ or $a<1$, then $\lim_{h\to 0} h^{a-1} \cdot \sin(h^{-c})$ does not converge.

(2) $a>1$, let $h\neq 0$, $0 \leq |h^{a-1} \cdot \sin(h^{-c})| \leq |h|^{a-1}$ and $\lim_{h\to 0} |h|^{a-1} = 0$ (choose $0<|h|<1$),

then $\lim_{h\to 0} |h^{a-1} \cdot \sin(h^{-c})| = 0 \Rightarrow \lim_{h\to 0} h^{a-1} \cdot \sin(h^{-c}) = 0$ exists. Thus, we have $a>1$.

By ③, ④, then $f'(0)$ exists $\Leftrightarrow$ $a>1$. $\blacksquare$

$$f'(x) = \begin{cases} a\cdot x^{a-1}\cdot \sin(x^{-c}) + x^{a}\cdot \cos(x^{-c})\cdot(-c)\cdot x^{-c-1}, & x \neq 0, \\ 0, & x=0. \quad (\text{since } (b)) \end{cases}$$

As $0<|x|\leq 1$, $|f'(x)| \leq |a||x|^{a-1} + c\cdot|x|^{a-c-1}$ and $|f'(0)|=0$.

(c)

⑤ Assume $a \geq 1+c \Rightarrow a-1-c \geq 0$ and $a>1$.

Then $|f'(x)| \leq |a|\cdot|x|^{a-1} + c\cdot|x|^{a-c-1} \leq |a|+c$ as $0 \leq |x| \leq 1$.

So f' is bounded.

⑥ Assume f' is bounded.

We know that $|f'(x)| \leq |a|\cdot|x|^{a-1} + c\cdot|x|^{a-c-1}$ as $0<|x|\leq 1$.

(1) $a \geq 1+c$, then $|f'(x)| \leq |a|+c \Rightarrow f'$ is bounded.

(2) $a < 1+c$, then $|f'(x)|$ is unbounded as $x \to 0^{+}$.

Thus, we have $a \geq 1+c$.

By ⑤, ⑥, then f' is bounded $\Leftrightarrow$ $a \geq 1+c$. $\blacksquare$

$$f'(x) = \begin{cases} a\cdot x^{a-1}\cdot \sin(x^{-c}) + x^{a}\cdot \cos(x^{-c})\cdot(-c)\cdot x^{-c-1}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(d) ⑦ Assume $a > 1+c \Rightarrow a-1 > c > 0$ and $a-1-c > 0$.

Then x^{a-1} and x^{a-1-c} are continuous for $x \neq 0$.

So $f'(x)$ is continuous for $x \neq 0$. — (☆☆☆)

Claim: $f'(x)$ is continuous at $x = 0$.

As $x \neq 0$, we have $0 \leq |f'(x)| \leq |a|\cdot|x|^{a-1} + c\cdot|x|^{a-1-c}$ and $0 < |x| \leq 1$.

Since $a-1 > c > 0$ and $a-1-c > 0$, then $\lim_{x\to 0} |x|^{a-1} = \lim_{x\to 0} |x|^{a-1-c} = 0$.

Thus, $\lim_{x\to 0} |f'(x)| = 0 \Rightarrow \lim_{x\to 0} f'(x) = 0 = f'(0)$.

So $f'(x)$ is continuous at $x = 0$ — (☆☆☆☆)

By (☆☆☆), (☆☆☆☆), then f' is continuous on $[-1, 1]$.

⑧ Assume f' is continuous on $[-1,1] \Rightarrow f'$ is continuous at $x = 0$.

$\lim_{x\to 0} f'(x) = f'(0) \Rightarrow \lim_{x\to 0} a\cdot x^{a-1}\cdot \sin(x^{-c}) + x^{a-1-c}\cdot(-c)\cdot\cos(x^{-c}) = 0$.

(1) $a = 1+c$, then $\lim_{x\to 0} a\cdot x^{a-1}\cdot\sin(x^{-c}) + x^{a-1-c}\cdot(-c)\cdot\cos(x^{-c})$

$= \lim_{x\to 0} a\cdot x^{c}\cdot \sin(x^{-c}) + (-c)\cdot\cos(x^{-c})$ does not converge.

(2) $a < 1+c$, then $\lim_{x\to 0} a\cdot x^{a-1}\cdot\sin(x^{-c}) + x^{a-1-c}\cdot(-c)\cdot\cos(x^{-c})$ does not converge.

(3) $a > 1+c$, let $x \neq 0$, $0 \leq |f'(x)| \leq |a| \cdot |x|^{a-1} + c \cdot |x|^{a-1-c}$ and $0 < |x| \leq 1$,

then $\lim_{x \to 0} |f'(x)| = 0 \Rightarrow \lim_{x \to 0} f'(x) = 0$, that is,

$$\lim_{x \to 0} a \cdot x^{a-1} \cdot \sin(x^{-c}) + x^{a-1-c} \cdot (-c) \cdot \cos(x^{-c}) = 0.$$

Thus, we have $a > 1+c$.

By ⑦, ⑧, then f' is continuous $\Leftrightarrow a > 1+c$. ■

6. Prove that (a) $\sin x \leq x$, $\forall x \geq 0$

(b) $x \leq \tan x$, $\forall x \in [0, \frac{\pi}{2}]$

(c) $x \geq \log x$, $\forall x > 0$

<pf>

(a) Let $f(x) = x - \sin x$, $\forall x \geq 0$.

$f(0) = 0 - \sin 0 = 0$, $f'(x) = 1 - \cos x \geq 0$ for any $x \geq 0$.

For any given $x \geq 0$, since f is continuous on $[0, x]$,

f is differentiable on $(0, x)$, by M.V.T., then $\exists\ 0 < t < x$ such that

$f(x) - f(0) = f'(t) \cdot (x - 0)$.

$\Rightarrow f(x) = f(x) - 0 = f'(t) \cdot x \geq 0$ for any $x \geq 0$

$\Rightarrow x \geq \sin x$ for any $x \geq 0$. $\blacksquare$

(b) Let $g(x) = \tan x - x$, $\forall x \in [0, \frac{\pi}{2}]$.

$g(0) = 0$, $g'(x) = \sec^2 x - 1 \geq 0$ as $0 \leq x \leq \frac{\pi}{2}$.

For any given $0 \leq x \leq \frac{\pi}{2}$, since g is continuous on $[0, x]$,

g is differentiable on $(0, x)$, by M.V.T., then $\exists\ 0 < t < x$ such that

$g(x) - g(0) = g'(t)(x - 0)$. $\Rightarrow$ $g(x) = g'(t) \cdot x \geq 0$ for any $0 \leq x \leq \frac{\pi}{2}$.

$\Rightarrow \tan x \geq x$ for any $0 \leq x \leq \frac{\pi}{2}$. $\blacksquare$

(C)

Let $h(x) = x - \log x$, $\forall x > 0$.

$h(1) = 1$, $h'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$ for any $x > 0$.

Then $h'(x) > 0$ as $x > 1$ and $h'(x) < 0$ as $0 < x < 1$.

① For any given $x > 1$, since h is continuous on $[1, x]$,

h is differentiable on $(1, x)$, by M.V.T., then $\exists\ 1 < t < x$ such that

$h(x) - h(1) = h'(t)(x-1)$.

$\Rightarrow h(x) - 1 = h'(t)(x-1) > 0$ for any $x > 1$.

$\Rightarrow h(x) > 1 > 0$ for any $x > 1 \Rightarrow x > \log x$ for any $x > 1$.

② For any given $0 < x < 1$, since h is continuous on $[x, 1]$,

h is differentiable on $(x, 1)$, by M.V.T., then $\exists\ x < t < 1$ such that

$h(1) - h(x) = h'(t)(1-x)$.

$\Rightarrow 1 - h(x) = h'(t)(1-x) < 0$ for any $0 < x < 1$.

$\Rightarrow h(x) > 1 > 0$ for any $0 < x < 1 \Rightarrow x > \log x$ for any $0 < x < 1$.

③ As $x = 1$, then $x = 1 > 0 = \log 1 = \log x$.

By ①,②,③, then $x > \log x$, $\forall x > 0$

$\Rightarrow x \geq \log x$, $\forall x > 0$.

7. Salas §3-2 ※ 24, 30, 38, 56

24.

$$f(x) = \frac{2x^2+x+1}{x^2+2x+1}$$

<sol>

$$f'(x) = \frac{(x^2+2x+1)\cdot(4x+1) - (2x^2+x+1)\cdot(2x+2)}{(x^2+2x+1)^2}$$

$$\Rightarrow f'(0) = -1 \quad \text{and} \quad f'(1) = \frac{20-16}{16} = \frac{1}{4}$$

※

30.

$$f(x) = h(x) + \frac{x}{h(x)} \qquad h(0)=3,\ h'(0)=2$$

<sol>

$$f'(x) = h'(x) + \frac{h(x) - x\cdot h'(x)}{h^2(x)}$$

$$\Rightarrow f'(0) = h'(0) + \frac{h(0) - 0\cdot h'(0)}{h^2(0)} = 2 + \frac{3-0}{9} = \frac{7}{3}$$

※

38. $f(x) = (x+2)(x^2-2x-8)$

<sol> $f'(x) = x^2-2x-8 + (x+2)(2x-2) = 3x^2-12$

Let $f'(x) = 0 \Rightarrow 3x^2 = 12 \Rightarrow x = 2$ or $x = -2$

$\therefore$ points are $(2,-32)$ or $(-2,0)$

※

5b

$p(x) = ax^3 + bx^2 + cx + d$

<sol>

$p'(x) = 3ax^2 + 2bx + c$

$\Rightarrow D = (2b)^2 - 4 \times 3a \times c = 4b^2 - 12ac$

(a) $D > 0 \iff p'$ has <u>two real</u> roots $\iff p$ has <u>two</u> horizontal tangents.

(b) $D = 0 \iff p'$ has <u>only one</u> real root $\iff p$ has <u>exactly</u> one horizontal tangent.

(c)

$D < 0 \iff p'$ has <u>no</u> real roots $\iff p$ has <u>no</u> horizontal tangent.

$\#$

8、Salas §3-5 ※ 20、24、66

20、

$$f(x) = [(6x+x^5)^{-1} + x]^2$$

<sol>

$$f'(x) = 2\cdot[(6x+x^5)^{-1}+x]\cdot\frac{d[(6x+x^5)^{-1}+x]}{dx}$$

$$= 2\cdot[(6x+x^5)^{-1}+x]\cdot\left\{\frac{d(6x+x^5)^{-1}}{dx}+1\right\}$$

$$= 2\cdot[(6x+x^5)^{-1}+x]\cdot\left\{-(6x+x^5)^{-2}\cdot(6+5x^4)+1\right\}$$

※

24、

$$y = u^3 - u + 1, \quad u = \frac{1-x}{1+x}$$

<sol>

$$\frac{dy}{dx} = \frac{du^3}{du}\frac{du}{dx} - \frac{du}{dx} + \frac{d(1)}{dx}$$

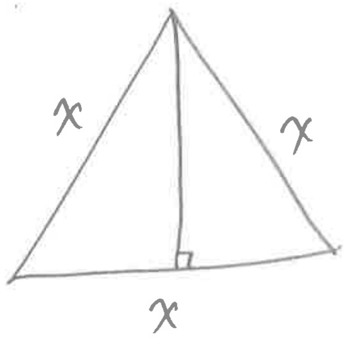
$$= 3u^2\cdot\frac{(1+x)\cdot(-1)-(1-x)\cdot 1}{(1+x)^2} - \frac{(1+x)\cdot(-1)-(1-x)\cdot 1}{(x+1)^2} + 0$$

$$= 3\cdot\left(\frac{1-x}{1+x}\right)^2\cdot\frac{-2}{(1+x)^2} + \frac{2}{(1+x)^2}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = -6+2 = -4$$

※

66.



$A = \frac{\sqrt{3}}{4}x^2$, $x = \frac{2\sqrt{3}}{3}h$

(area)

<sol>

$$\frac{dA}{dh} = \frac{dA}{dx}\frac{dx}{dh}$$

$$= \frac{\sqrt{3}}{4} \cdot 2x \cdot \frac{2\sqrt{3}}{3} = x = \frac{2\sqrt{3}}{3}h$$

$$\Rightarrow \left.\frac{dA}{dh}\right|_{h=2\sqrt{3}} = \frac{2\sqrt{3}}{3} \cdot 2\sqrt{3} = 4$$

※

9. Salas §3-6 : ※28, 40, 50

28、

<Sol> $\frac{d}{dt}\left[t\cdot\frac{d}{dt}(\cos t^2)\right]$

$= \frac{d}{dt}\left[t\cdot(-\sin t^2)\cdot 2t\right]$

$= \frac{d}{dt}\left[-2t^2\cdot\sin t^2\right]$

$= -4t\sin t^2 + (-2t^2)\cdot\cos t^2\cdot 2t$

$= -4t\sin t^2 - 4t^3\cdot\cos t^2$ ※

40、 $0\le x\le 2\pi$

<Sol> $y=\cos x-\sqrt{3}\sin x$

$y' = -\sin x-\sqrt{3}\cos x$

Let $y'=0$. $\Rightarrow$ $\sin x = -\sqrt{3}\cos x$

$\Rightarrow$ $\tan x = -\sqrt{3}$, $0\le x\le 2\pi$

$\Rightarrow$ $x=\frac{2\pi}{3}, \frac{5\pi}{3}$ ※

50、 $0<x<2\pi$, $f(x)=\sin x-\cos x$

<sol> $f'(x)=\cos x+\sin x$

(a)

$f'(x)=0 \Rightarrow \cos x=-\sin x \Rightarrow \tan x=-1 \Rightarrow x=\frac{3\pi}{4}, \frac{7\pi}{4}$

#

(b)

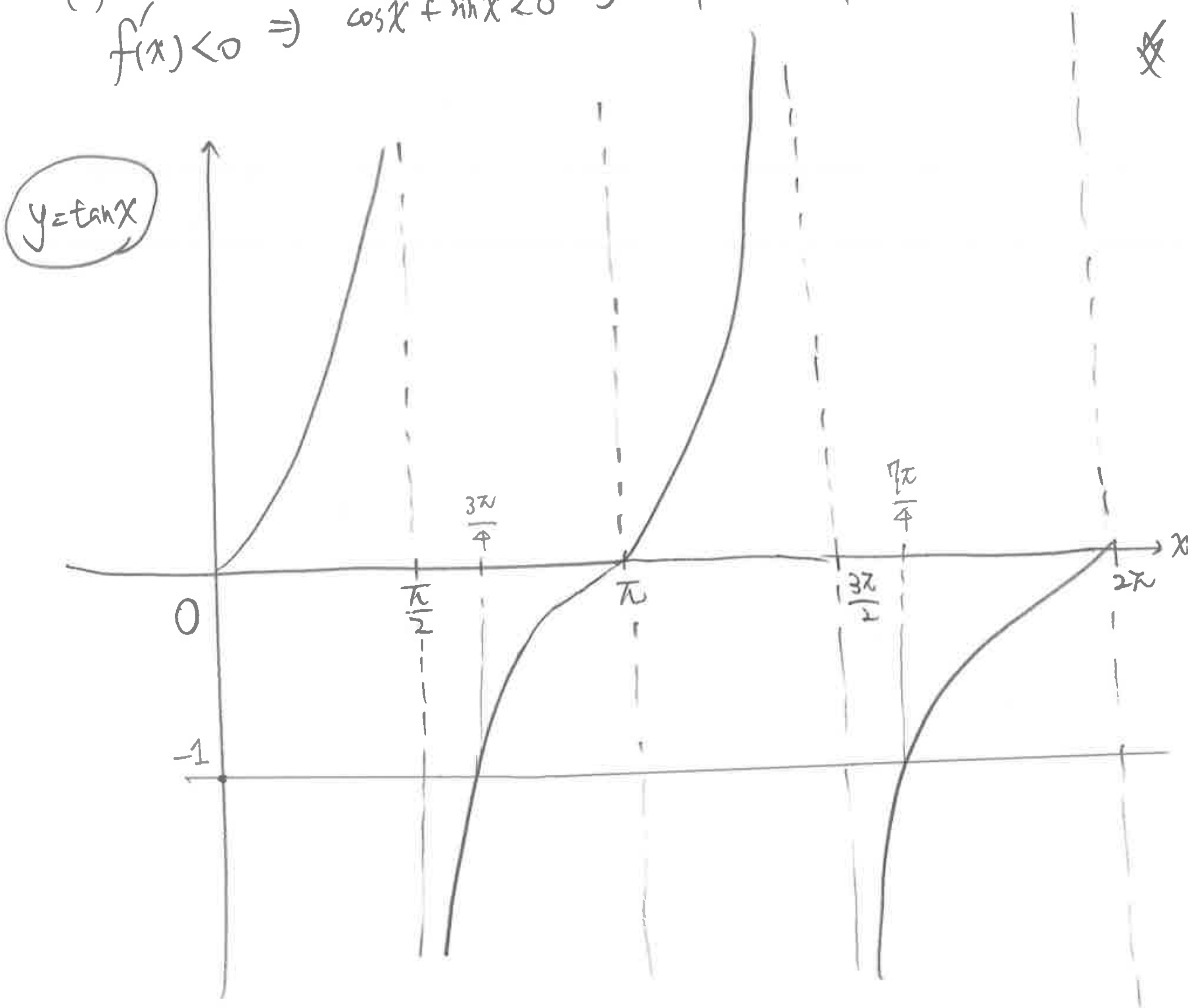
$f'(x)>0 \Rightarrow \cos x+\sin x>0 \Rightarrow 0<x<\frac{3\pi}{4}$ or $\frac{7\pi}{4}<x<2\pi$

#

(c)

$f'(x)<0 \Rightarrow \cos x+\sin x<0 \Rightarrow \frac{3\pi}{4}<x<\frac{7\pi}{4}$

#



10. Salas : §4-1 : ※ 6, 13, 15, 20

6.

$$f(x) = 3\sqrt{x} - 4x, \quad 1 \le x \le 4$$

<sol>

$$f'(x) = \frac{3}{2\sqrt{x}} - 4$$

Let $\dfrac{f(4) - f(1)}{4-1} = \dfrac{-10-(-1)}{3} = -3$. $\Rightarrow \dfrac{3}{2\sqrt{x}} - 4 = -3$

$$\Rightarrow \frac{3}{2\sqrt{x}} = 1 \Rightarrow \sqrt{x} = \frac{3}{2} \Rightarrow x = \frac{9}{4}$$

13. $\exists f(x)$ satisfies

$$f(0) = 2, \quad f(2) = 5, \quad f'(x) \le 1 \quad \forall x \in (0,2) \ ?$$

<sol> No !!

By M.V.T., then $\dfrac{f(2) - f(0)}{2-0} = \dfrac{5-2}{2} = \underbrace{\dfrac{3}{2} = f'(c)}_{>1.}$ for some $0 < c < 2$.

15. f is continuous on $[2,6]$, f is differentiable on $(2,6)$, $1 \le f'(x) \le 3 \ \forall x \in (2,6)$,

show that $4 \le f(6) - f(2) \le 12$.

<pf>

By M.V.T., then $\dfrac{f(6) - f(2)}{6-2} = f'(c)$ for some $2 < c < 6$.

$$\Rightarrow 1 \le \frac{f(6) - f(2)}{6-2} \le 3$$

$$\Rightarrow 4 \le f(6) - f(2) \le 12 \qquad \blacksquare$$

20. $f(x) = \dfrac{1}{x}$, $a=-1$, $b=1$

<pf>

$$f'(c) = \frac{f(1) - f(-1)}{1-(-1)} = \frac{2}{2} = 1, \text{ but } f'(x) = \frac{-1}{x^2} < 0, \ \forall x \in \mathbb{R}.$$

$\Rightarrow$ there is no number c for which $f'(c) = 1$ and f is not continuous at 0.

($\because$ f is not differentiable at 0.) $\blacksquare$

11、salas §4-2：※ 10、19、35、56

10、

$$f(x)=\frac{x}{1+x^2}$$

<sol>

$$f'(x)=\frac{1+x^2-x\cdot 2x}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}$$

$f'(x)=0 \Rightarrow x=\pm 1$

① $f'(x)\geq 0 \iff -1\leq x\leq 1$ ∴ f is increasing on $[-1,1]$

② $f'(x)\leq 0 \iff x\geq 1$ or $x\leq -1$ ∴ f is decreasing on $(-\infty,-1]$, $[1,\infty)$

※

19、

$f(x)=x-\cos x$, $0\leq x\leq 2\pi$

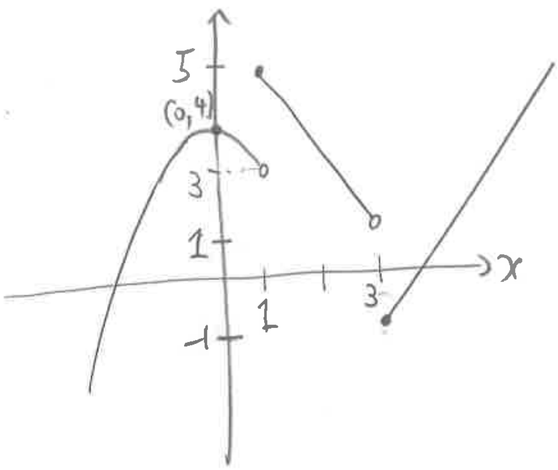
<sol>

$f'(x)=1+\sin x\geq 0$ ∴ f is increasing on $[0,2\pi]$.

※

35.

$$f(x) = \begin{cases} 4-x^2, & x<1 \\ 7-2x, & 1 \le x < 3 \\ 3x-10, & 3 \le x \end{cases} \Rightarrow f'(x) = \begin{cases} -2x, & x<1 \\ -2, & 1<x<3 \\ 3, & 3<x \end{cases}$$



$\Rightarrow$ f is increasing on $(-\infty, 0]$, $[3, \infty)$ ※

f is decreasing on $[0, 1)$, $[1, 3]$ ※

56. f, g are differentiable on $(-c, c)$, $f(0) = g(0)$,

(a) if $f'(x) > g'(x)$ $\forall x \in (0, c) \Rightarrow f(x) > g(x)$; $\forall x \in (0, c)$

(b) if $f'(x) > g'(x)$ $\forall x \in (-c, 0) \Rightarrow f(x) < g(x)$; $\forall x \in (-c, 0)$

<pf> Let $h(x) = f(x) - g(x)$. $h'(x) = f'(x) - g'(x)$

(a)

Since $h'(x) > 0$ $\forall x \in (0, c)$, so h is increasing on $(0, c)$. $h(0) = 0$.

$\Rightarrow h(x) > h(0) = 0, \forall c > x > 0$

$\Rightarrow f(x) > g(x)$ $\forall c > x > 0$. ∎

(b)

Since $h'(x) < 0$ $\forall x \in (0, c)$, so h is decreasing on $(-c, 0)$. $h(0) = 0$.

$\Rightarrow h(x) < h(0) = 0, \forall -c < x < 0 \Rightarrow f(x) < g(x)$, $\forall -c < x < 0$. ∎