NOTES ON $\limsup$ AND $\liminf$

Here, I provide a detailed explanation on the consequences of $\limsup_k a_k < \alpha$ and $\limsup_k a_k > \alpha$. There are, of course, analogous statements for $\liminf$.

Proposition 0.1.

$$\limsup_k a_k < \alpha \Rightarrow \textit{There exists } N \in \mathbb{N} \textit{ such that } a_k < \alpha \ \forall k \geq N.$$

Proof. Since $\limsup_k a_k < \alpha$, there exists $\beta < \alpha$ so that $\limsup_k a_k < \beta < \alpha$. It is then sufficient to show that a_k is eventually $\leq \beta$. That is, there is $N \in \mathbb{N}$ so that $a_k \leq \beta \ \forall k \geq N$. If not, for each $n \in \mathbb{N}$, there exists a_{k_n} so that $a_{k_n} \geq \beta$. All these a_{k_n}'s form a subsequence of $\{a_k\}$. Since $\limsup_k a_k \leq \alpha$, for large enough n, we have

$$\beta \leq a_{k_n} \leq \alpha$$

and without loss of generality we may assume the bounds are true for all n. By Bolzano-Weirstrass Theorem, a_{k_n} (and so a_k) has a convergent subsequence whose limit is no less than β. This implies that $\limsup_k a_k \geq \beta$, which contradicts the fact $\limsup_k a_k < \beta$.

□

The converse of the proposition is false. Take, for example, $a_k = 1 - \frac{1}{k}$. Then $a_k < 1 \ \forall k$, but $\limsup_k a_k = \lim_k a_k = 1 \not< 1$. The partial converse of Proposition 0.1 is the following:

Proposition 0.2.

$$\limsup_k a_k > \alpha \Rightarrow \ \textit{There exists a subsequence } \{a_{k'}\} \subset \{a_k\} \textit{ such that } a_{k'} > \alpha \ \forall k'.$$

Proof. Pick β so that $\alpha < \beta < \limsup_k a_k$. By definition of $\limsup$, there is a convergent subsequence $\{a_{k'}\} \subset \{a_k\}$ so that $a_{k'} \to a \in (\beta, \limsup_k a_k)$ as $k' \to \infty$. Pick ϵ small enough so that $\beta - \epsilon > \alpha$. Since $a_{k'} \to a$, there exists $N \in \mathbb{N}$ so that $a_{k'} > a - \epsilon > \beta - \epsilon > \alpha$ for all $k' \geq N$. The subsequence $\{a_{k'}\}_{k' \geq N}$ is the desired subsequence.

□

The analogous statements for $\liminf$, which are proved vice versa, are the followings.

Proposition 0.3.

$$\liminf_k a_k > \alpha \Rightarrow \textit{There exists } N \in \mathbb{N} \textit{ such that } a_k > \alpha \ \forall k \geq N.$$

Proposition 0.4.
$\liminf_{k} a_k < \alpha \Rightarrow$ *There exists a subsequence* $\{a_{k'}\} \subset \{a_k\}$ *such that* $a_{k'} < \alpha \ \forall k'$.

Of course, all propositions above can be stated equivalently with their contrapositives.

Proposition 0.5.
There exists a subsequence $\{a_{k'}\} \subset \{a_k\}$ *such that* $a_{k'} \geq \alpha \ \forall k' \Rightarrow \limsup_{k} a_k \geq \alpha$.

Proposition 0.6.
$$\text{There exists } N \in \mathbb{N} \text{ such that } a_k \leq \alpha \ \forall k \geq N \Rightarrow \limsup_{k} a_k \leq \alpha.$$

Proposition 0.7.
There exists a subsequence $\{a_{k'}\} \subset \{a_k\}$ *such that* $a_{k'} \leq \alpha \ \forall k' \Rightarrow \liminf_{k} a_k \leq \alpha$.

Proposition 0.8.
$$\text{There exists } N \in \mathbb{N} \text{ such that } a_k \geq \alpha \ \forall k \geq N \Rightarrow \liminf_{k} a_k \geq \alpha.$$