Name and Student ID's: ____________________

Homework 1, Advanced Calculus 1

1. Prove Proposition 1.15 of Rudin (page 7).

2. In class we have defined the rational number $\mathbb{Q}$ as the set of equivalence classes:

$$\mathbb{Q} := (\mathbb{Z} \times \mathbb{Z}\backslash\{0\})/\sim,$$

where $(a, b) \sim (c, d) \Leftrightarrow ad = bc$. Define field operations on $\mathbb{Q}$ by

- $[(a, b)] + [(c, d)] = [(ad + bc, bd)]$.
- $[(a, b)] \cdot [(c, d)] = [(ac, bd)]$.

(a) Show that the operations above are well defined. That is, different choices of representative from the classes give the same equivalence classes on the right hand side.

(b) Prove that $\mathbb{Q}$ with these operations is a field. You do not need to show associativity and commutativity in both operations.

3. Rudin Exercise 6 ab.

4. Rudin Exercise 6 cd.

5. Rudin Exercise 7 abcd.

6. Rudin Exercise 7efg.

The next two problems deal with the *decimal expansion* of real numbers.

7. Given a real number $x > 0$,

(a) prove that there is a *largest* integer $n_0 \leq x$. (Use Archimedean property)

(b) Inductively, for each $k \in \mathbb{N}$, let n_k be the largest integer so that

$$n_k \leq 10^k \left(x - n_0 - n_1 10^{-1} - \cdots - n_{k-1} 10^{-(k-1)}\right),$$

or equivalently

$$A_k = \sum_{j=0}^{k} n_j 10^{-j} \leq x.$$

Show that $0 \leq n_j \leq 9$ for all $j > 1$.

(c) Prove that the sequence $E = \{A_k\}$ is monotonic and bounded above, and therefore $\lim_k A_k$ exists and is equal to $\sup E$.

8. (a) Prove that $x = \sup E$.

   (b) Eliminating sequences $\{n_j\}$ mentioned above with the property that $n_j = 9$ for all j after a certain term (which is impossible from its construction anyway), prove that

$$\sum_{j=0}^{\infty} n_j 10^{-j} = \sum_{j=0}^{\infty} m_j 10^{-j} \Rightarrow n_j = m_j \;\; \forall j.$$

We have shown that every positive real number may be uniquely expressed by an infinite sequence of integers $\{n_j\}$ with $0 \leq n_j \leq 9 \;\; \forall j > 1$ so that

$$x = \sum_{j=0}^{\infty} n_j 10^{-j}.$$

We usually denote it by

$$x = n_0.n_1 n_2 n_3 \cdots ,$$

and call it the *decimal expansion* of x. Note that we may replace 10 by any other positive integer $N > 1$ and the entire construction holds without any major modification (you may be familiar with the expansion with $N = 2$).