

Hw b.

Rudin, ch7, *18.

Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t) dt, \quad a \leq x \leq b.$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

(P)

① Since $\{f_n\}$ is uniformly bounded, then $|f_n(x)| \leq M \quad \forall x \in [a, b] \quad \text{then} \quad M \text{ for some } M \in \mathbb{R}$.

$$\Rightarrow |F_n(x)| \leq \int_a^x |f_n(t)| dt \leq M(x-a) \leq M(b-a) \quad \forall x \in [a, b] \quad \forall n \in \mathbb{N}.$$

So $\{F_n\}$ is uniformly bounded.

②

Given $\varepsilon > 0$, $\exists \delta = \frac{\varepsilon}{M} > 0$, such that if $|x-y| < \delta$, then we have

$$\begin{aligned} |F_n(x) - F_n(y)| &= \left| \int_a^x f_n(t) dt - \int_a^y f_n(t) dt \right| \\ &= \left| \int_y^x f_n(t) dt \right| \leq M|x-y| < M\delta = \varepsilon. \end{aligned}$$

So $\{F_n\}$ is equicontinuous.

By Theorem 7.5 (A is s.t. chm), ①+②, then there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$. ■

Rudin, ch 7, §19.

2, K : compact metric space, $S \subseteq (C(K), \| \cdot \|)$ where $\|f\| = \sup_{x \in K} |f(x)|$.

Prove that $(S, \| \cdot \|)$ is compact $\Leftrightarrow S$ is uniformly closed, pointwise bounded, and equicontinuous.

\leftarrow Let S be uniformly closed, pointwise bounded, and equicontinuous.
Let $\{g_n\}$ be a sequence of S .

" If K is compact, $\{g_n\} \subseteq S \subseteq C(K)$, then $\{g_n\}$ is pointwise bounded, equicontinuous on K .

By thm 7.25 (Ascoli's thm), then $\{g_n\}$ has a uniformly convergent subsequence.

Let $g_{n_k} \rightarrow g \in C(X)$ as $k \rightarrow \infty$.

" If S is closed, then $g \in S$.

By Ch 2, §26, that is, S is sequentially compact, then S is compact.

\Rightarrow Let S be compact.

Then S is closed and bounded.

① " If S is bounded, then $\|f\| \leq M \ \forall f \in S$ for some $M \in \mathbb{R}$.

$\Rightarrow S$ is uniformly bounded.

②

" If S is closed set of $C(K)$, then S is called uniformly closed.
 K is compact

③ Claim: S is equicontinuous, that is, given $\varepsilon > 0$, $\exists \delta > 0$, if $d(x, y) < \delta$, then

$$|g(x) - g(y)| < \varepsilon \text{ for any } g \in S.$$

Suppose S is not equicontinuous, that is, $\exists \varepsilon > 0$, $\forall \delta > 0$, if $d(x, y) < \delta$, then

$$|g(x) - g(y)| \geq \varepsilon \text{ for some } g \in S.$$

Let $\delta = \frac{1}{n} > 0$, choose $x_n, y_n \in K$ such that $d(x_n, y_n) < \frac{1}{n}$, then $|g_n(x_n) - g_n(y_n)| \geq \varepsilon$ for some $g_n \in S$.

Thus, we a sequence $\{g_n\} \subseteq C(K)$, K is compact.

Let $\{g_{n_k}\}$ be any sequence of $\{g_n\}$.

Then $|g_{n_k}(x_{n_k}) - g_{n_k}(y_{n_k})| \geq \varepsilon$ for all $k \in \mathbb{N}$, so $\{g_{n_k}\}$ is also not equicontinuous.

By Thm 1.24, then $\{g_{n_k}\}$ does not converge in $C(K)$.

$\Rightarrow S$ is not compact (contradiction)

Thus, S is equicontinuous. □

3, fudm, ch7, 20

f is continuous on $[0,1]$, $\int_0^1 f(x) \cdot x^n dx = 0$, $n=0,1,2,\dots$.

Prove that $f(x)=0$ on $[0,1]$.

(pf)

① Claim: $\int_0^1 f(x)^2 dx = 0$.

Since f is continuous on $[0,1]$, by Thm 7.26 (Weierstrass thm),

then \exists polynomials p_n such that $\lim_{n \rightarrow \infty} p_n(x) = f(x)$ uniformly on $[0,1]$.

$\because [0,1]$ is compact $\therefore f$ is bounded.

$\because \{p_n\}$ is convergent and f is bounded, then $\{p_n\}$ is uniformly bounded.

$\because p_n \xrightarrow[\text{as } n \rightarrow \infty]{\text{uniformly}} f$, f is bounded, then $p_n f \xrightarrow[\text{as } n \rightarrow \infty]{\text{uniformly}} f^2$.

$\because p_n f \in R([0,1])$, $p_n f \xrightarrow[n \rightarrow \infty]{\text{uniformly}} f^2$, by Thm 7.16,

then $\lim_{n \rightarrow \infty} \int_0^1 p_n(x) f(x) dx = \int_0^1 f(x)^2 dx$ and $f^2 \in R([0,1])$.

② By our assumption, then $\int_0^1 p_n(x) f(x) dx = 0$ for all $n \in \mathbb{N}$.

$$\Rightarrow \int_0^1 f(x)^2 dx = 0.$$

③ $\because f^2 \geq 0$, f^2 is continuous on $[0,1]$, $\int_0^1 f(x)^2 dx = 0$, then $f(x)^2 = 0$ on $[0,1]$
 $\Rightarrow f(x) = 0$ on $[0,1]$.

4. Rudin ch 7, *22.

$f \in R(\alpha)$ on $[a, b]$.

Prove that there are polynomials p_n such that $\lim_{n \rightarrow \infty} \int_a^b |f - p_n|^2 d\alpha = 0$.

(pf)

① By Ex 12, ch 6, given $\epsilon = \frac{1}{n} > 0$, then \exists continuous function f_n on $[a, b]$ such that

$$\lim_{n \rightarrow \infty} \int_a^b |f - f_n|^2 d\alpha = 0.$$

② By thm 7.26, for each $n \in \mathbb{N}$, f_n is continuous on $[a, b]$, given $\epsilon = \frac{1}{n} > 0$,

then \exists polynomial p_n such that $|p_n(x) - f_n(x)| < \frac{1}{n} \quad \forall x \in [a, b]$.

③ By Ex 11, ch 6, then $\|f - p_n\|_2 \leq \|f - f_n\|_2 + \|f_n - p_n\|_2$

$$\lim_{n \rightarrow \infty} \|f - f_n\|_2 = \lim_{n \rightarrow \infty} \left(\int_a^b |f - f_n|^2 d\alpha \right)^{\frac{1}{2}} = \left(\lim_{n \rightarrow \infty} \int_a^b |f - f_n|^2 d\alpha \right)^{\frac{1}{2}} = 0$$

$$\lim_{n \rightarrow \infty} \|p_n - f_n\|_2 = \lim_{n \rightarrow \infty} \left(\int_a^b |p_n - f_n|^2 d\alpha \right)^{\frac{1}{2}} \leq \lim_{n \rightarrow \infty} \frac{\sqrt{b-a}}{n} = 0 \Rightarrow \lim_{n \rightarrow \infty} \|p_n - f_n\|_2 = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \|f - p_n\|_2 \leq \lim_{n \rightarrow \infty} (\|f - f_n\|_2 + \|f_n - p_n\|_2) = 0 \Rightarrow \lim_{n \rightarrow \infty} \|f - p_n\|_2 = 0.$$

$$\Rightarrow \lim_{n \rightarrow \infty} \int_a^b |f - p_n|^2 d\alpha = 0.$$



HW 10.

5. Thm 7.31: Suppose \mathcal{A} is an algebra of functions on a set E , \mathcal{A} separates points on E , and \mathcal{A} vanishes at no point of E . Suppose x_1, x_2 are distinct points of E , and c_1, c_2 are constant (real if \mathcal{A} is a real algebra). Then \mathcal{A} contains a function f such that $f(x_1) = c_1, f(x_2) = c_2$.

- If for any $x_1 \neq x_2 \in E$, there is a function $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$, then \mathcal{A} is called "separate points on E ".
- If for any $x \in E$, there is a function $f \in \mathcal{A}$ such that $f(x) \neq 0$, then \mathcal{A} is called "vanishes at no point of E ".

$\langle pf \rangle$

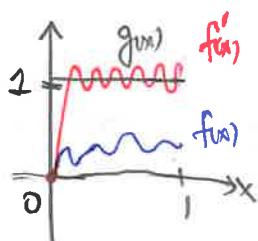
① \mathcal{A} : separate points on E , does not vanish at no point of E , that is, $\exists x_0 \in E, \forall f \in \mathcal{A}$ such that $f(x_0) = 0$.

Given $x \in E, x \neq x_0$. Then $f_x(x) \neq f_x(x_0) = 0$ for some $f_x \in \mathcal{A}$.

Let $E = [0, 1], x_0 = 0$.

Let $\mathcal{A} = \{f \in C([0, 1]) \mid f(0) = 0, f(x) \neq 0, 0 < x \leq 1\}$

Suppose Thm 7.31 holds. Then $\overline{\mathcal{A}} = C([0, 1])$.



Choose $g(x) = 1, 0 \leq x \leq 1$. Then $g \in C([0, 1]) = \overline{\mathcal{A}}$.

Given $\varepsilon = \frac{1}{2} > 0, \exists f' \in \mathcal{A}$ with $f'(0) = 0, f'(x) > 0, f'$ is continuous such that

$\|f' - g\| < \frac{1}{2}$. But we know $\|f - g\| = \sup_{x \in [0, 1]} |f(x) - g(x)| \geq 1$. (contradiction)

So Thm 7.31 false.



(2)

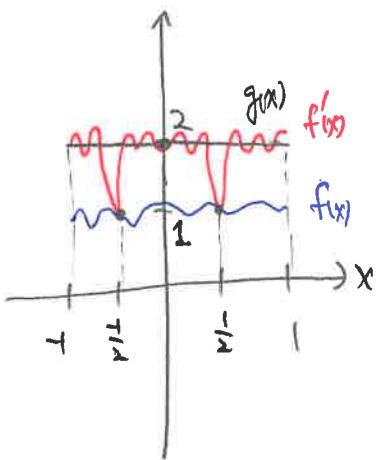
\mathbb{A} vanishes at no point of E , does not separate points on E ,
 that is, $\exists x_0, y_0 \in E, x_0 \neq y_0, \forall f \in \mathbb{A} \text{ such that } f(x_0) = f(y_0)$.

Given $x \in E, \exists f \in \mathbb{A} \text{ s.t. } f(x) \neq 0 \Rightarrow f(x_0) = f(y_0) \neq 0 \forall f \in \mathbb{A}$.

Let $E = [-1, 1], x_0 = \frac{1}{2}, y_0 = -\frac{1}{2}$.

Let $\mathbb{A} = \{f \in C([-1, 1]) \mid f(\frac{1}{2}) = f(-\frac{1}{2}) = 1, f(x) \neq 0, -1 \leq x \leq 1\}$.

Suppose Theorem 2.31 holds. Then $\overline{\mathbb{A}} = C([-1, 1])$.



Choose $g(x) = 2, -1 \leq x \leq 1$. Then $g \in C([-1, 1]) = \overline{\mathbb{A}}$.

Given $\varepsilon = \frac{1}{2} > 0, \exists f' \in \mathbb{A}, f'(\frac{1}{2}) = f'(-\frac{1}{2}) = 1, f'(x) > 0, f' \text{ is continuous}$,
 such that $\|f' - g\| < \varepsilon = \frac{1}{2}$.

But we know $\|f' - g\| = \sup_{x \in [-1, 1]} |f'(x) - g(x)| \geq 1$. (contradiction)

So Theorem 2.31 false.

6. Rudin, ch 7, #24.

Let X be a metric space with metric d . Fix a point $a \in X$. Assign to each $p \in X$ the

function f_p defined by $f_p(x) = d(a, p) - d(x, a)$, $\forall x \in X$.

Prove that $|f_p(x)| \leq d(a, p)$ $\forall x \in X$ and $f_p \in C(X)$.

Prove that $\|f_p - f_q\| = d(p, q)$ $\forall p, q \in X$.

(f)

$$\textcircled{1} \quad |f_p(x)| = |d(x, p) - d(x, a)|, \quad \forall x \in X.$$

$$d(x, p) \leq d(x, a) + d(a, p) \Rightarrow d(x, p) - d(x, a) \leq d(a, p)$$

$$d(x, a) \leq d(x, p) + d(p, a) \Rightarrow d(x, a) - d(x, p) \leq d(p, a) = d(a, p)$$

$$\Rightarrow -d(a, p) \leq d(x, p) - d(x, a) \leq d(a, p) \Rightarrow |d(x, p) - d(x, a)| \leq d(a, p).$$

$$\therefore |f_p(x)| = |d(x, p) - d(x, a)| \leq d(a, p) \quad \forall x \in X.$$

$$\textcircled{2} \quad \forall x, y \in X, \text{ then } |f_p(x) - f_p(y)| = |d(x, p) - d(x, a) - d(y, p) + d(y, a)| \\ \leq |d(x, p) - d(y, p)| + |d(y, a) - d(x, a)|, \quad \text{by } \textcircled{1}$$
$$\leq d(x, y) + d(x, y) = 2d(x, y)$$

Given $\varepsilon > 0$, $\exists \delta = \frac{\varepsilon}{2} > 0$, s.t. if $d(x, y) < \delta$,

$$\text{then } |f_p(x) - f_p(y)| \leq 2d(x, y) < 2\delta = \varepsilon.$$

So f_p is continuous on X .

③

$$\|f_p - f_q\| = \sup_{x \in X} |f_p(x) - f_q(x)|, \quad \forall p, q \in X.$$

$$\begin{aligned} |f_p(x) - f_q(x)| &= |d(x, p) - d(x, q) - d(x, q) + d(x, q)| \\ &= |d(x, p) - d(x, q)| \leq d(p, q), \quad \forall x \in E \end{aligned}$$

$$\Rightarrow \sup_{x \in X} |f_p(x) - f_q(x)| \leq d(p, q).$$

Choose $x = p$ or $x = q$, then $|f_p(q) - f_q(p)| = |d(p, q)| = d(p, q)$

or $|f_p(q) - f_q(q)| = |d(q, p)| = d(p, q)$.

So $\|f_p - f_q\| = \sup_{x \in X} |f_p(x) - f_q(x)| = d(p, q)$.



$\langle pf \rangle$ Let $\Phi = (X, d) \rightarrow (C(X), \|\cdot\|)$ be defined by $\Phi(p) = f_p$.

$\forall p, q \in X$, then we have $d(p, q) = \|f_p - f_q\|$. (\because ③)

$\Rightarrow \Phi$ is Isometry.



7. Rudin, ch 7, #24,

Let Y be the closure of $\bar{\Phi}(X)$ in $C(X)$.

Show that Y is complete.

(pf)

Let $\{x_n\}$ be a Cauchy sequence in Y .

By Thm 7.15, then $C(X)$ is complete.

$\Rightarrow \{x_n\}$ is a convergent sequence in $C(X)$.

Let $x_n \rightarrow x \in X$ as $n \rightarrow \infty$,

Since Y is the closure of $\bar{\Phi}(X) = \{ \Phi(p) = f_p \mid p \in X \} \subseteq C(X)$,

then Y is closed.

$\Rightarrow x \in Y$.

So $x_n \rightarrow x \in Y$ as $n \rightarrow \infty$.

Thus, Y is complete. ■

