Name and Student ID's: ____

Homework 11, Advanced Calculus 1

1. Rudin Chapter 8 Exercise 1

Solution: By induction on n, we can readily compute that for each n,

$$f^{(n)}(x) = \frac{P_n(x)}{x^{3n}} e^{\frac{-1}{x^2}},$$

where P_n is a polynomial for all $x \neq 0$ (do it!).

Then we apply induction again to show that $f^{(n)}(0) = 0$ for all n, where each step uses the following L'Hospital computation with $k \in \mathbb{N}$:

$$\lim_{x\to 0} \frac{e^{\frac{-1}{x^2}}}{x^k} = \lim_{x\to 0} \frac{x^{-k}}{e^{\frac{-1}{x^2}}} = \frac{k}{2} \lim_{x\to 0} \frac{x^{-k+2}}{e^{\frac{-1}{x^2}}} = \cdots = 0$$

For $n = 1$, we have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{e^{\frac{-1}{x^2}}}{x} = 0.$$

Suppose that $f^{(n)}(0) = 0$, then

$$\lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x} = P_n(0) \lim_{x\to 0} \frac{e^{\frac{-1}{x^2}}}{x^{3n+1}} = 0$$

and therefore $f^{(n+1)}(0) = 0$ and the induction is completed.

2. Rudin Chapter 8 Exercise 9a

Solution: Let $a_N = s_N - \log N$. We show that it is a bounded and monotonic sequence and therefore convergent.

Since $\log N = \int_1^N \frac{1}{x}\, dx$, or the area under the curve $f(x) = \frac{1}{x}$ from 1 to N. The number s_N can be seen as the sum of areas of rectangles of base 1 and height $\frac{1}{n}$, $1 \leq n \leq N$. In fact, it is $U(f, P_N)$ on $[1, N]$, where P_N is the partition of N-equally subdivisions. One may see that for the same partition, $L(f, P_N) = s_N - 1$. Both of these follow from the fact that f is a decreasing function and so the supremum (infimum) of f occurs at left (right) endpoint.Therefore, $\log N \leq s_N$ and $\log N \geq s_N - 1$ and we have

$$0 \leq s_N - \log N \leq 1.$$

That is, a_N is bounded. For monotonicity, we have, for all N,

$$a_{N+1} - a_N = \frac{1}{N+1} - \int_N^{N+1} \frac{1}{x}\, dx \leq \frac{1}{N+1} - \frac{1}{N+1} = 0$$

since again $\frac{1}{x}$ is decreasing.

3. Rudin Chapter 8 Exercise 23

Solution: The curve $\gamma(t)$ in this problem may be written in the polar form

$$\gamma(t) = r(t)e^{i\theta(t)}$$

where $r(t) > 0$ for all $t \in [a, b]$ and $\theta(t) \in \mathbb{R}$. Since $\gamma(a) = \gamma(b)$, it follows that $r(a) = r(b)$ and $\theta(b) = \theta(a) + 2k\pi$ for some $k \in \mathbb{Z}$. We will show that $Ind(\gamma) = k$.

With $\gamma(t) \neq 0\ \forall t$, we have

$$\frac{\gamma'(t)}{\gamma(t)} = \frac{r'(t)}{r(t)} - i\theta'(t) = \frac{d}{dt}(\log(r(t)) - i\theta(t)).$$

Then, by fundamental theorem of calculus (for complex valued function) and discussion above, we have

$$\frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)}\,dt = \theta(b) - \theta(a) = k := Ind(\gamma) \in \mathbb{Z}.$$

For $[a, b] = [0, 2\pi]$, we have $\theta(t) = t$ and it is clear that $\theta(2\pi) - \theta(0) = 2\pi$, or $Ind(\gamma) = 1$.

The quantity is called winding number since it counts how many circles around origin (or a loop containing origin that can be continuously deformed into a circle) the curve γ has travelled during the time $[a, b]$ with signs ($+1$ for counterclockwise loop and -1 for clockwise).

4. Rudin Chapter 8 Exercise 24

Solution: Intuitively, if γ does not intersect negative real axis, it can never travel a complete loop and therefore the winding number has to be 0. Precisely, for $c \in [0, \infty)$, the closed curve

$$\gamma_c(t) := \gamma(t) + c$$

is never zero (otherwise $\gamma(t) = -c$ is on the negative real axis). Also, since $\gamma_c' = \gamma$, we have

$$Ind(\gamma_c) = \frac{1}{2\pi i}\int_a^b \frac{\gamma(t)}{\gamma(t) + c}\,dt$$

which tends to 0 as $c \to \infty$. However, since the expression above is continuous in c (integral of a continuous function) and integer valued, it must be constant. It then follows that $Ind(\gamma) = Ind(\gamma_c) = 0$ for all $c \geq 0$.

5. Rudin Chapter 8 Exercise 25

Solution: Following hint, let $\gamma = \frac{\gamma_2}{\gamma_1}$, then

$$|1 - \gamma(t)| = |\frac{\gamma_1(t) - \gamma_2(t)}{\gamma_1(t)}| < 1$$

since $|\gamma_1(t)-\gamma_2(t)| < |\gamma_1(t)|$ for all t. Therefore, $\gamma(t)$ never lies on negative real axia and by previous problem, $Ind(\gamma) = 0$.

On the other hand, straightforward computations show that

$$\frac{\gamma'}{\gamma} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1}$$

and therefore $Ind(\gamma) = Ind(\gamma_2) - Ind(\gamma_1) = 0$. The result then follows.

6. Rudin Chapter 8 Exercise 26

Solution: Take trigonometric polynomials P_1, P_2 as in the hint. By triangle inequality, we have $|\gamma(t)| - |P_1(t)| \leq |\gamma(t) - P_1(t)| \leq \frac{\delta}{4}$, or $|P_1(t)| \geq |\gamma(t)| - \frac{\delta}{4} \geq \frac{3\delta}{4}$.

Also, we have $|P_1(t) - P_2(t)| \leq |P_1(t) - \gamma(t)| + |\gamma(t) - P_2(t)| \leq \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}$. The two inequalities we have so far imply

$$|P_1(t) - P_2(t)| < |P_1(t)| \quad \forall t$$

and therefore by Problem 5, $Ind(P_1) = Ind(P_2)$. We call the common value $Ind(\gamma)$.

Since both P_1 and P_2 have norms $\geq \frac{3\delta}{4} > 0$, they will never be 0. Moreover, $P_1(a) = P_1(b) = P_2(a) = P_2(b)$ by Theorem 8.15 (with $[a, b]$ rescaled and translated to $[0, 2\pi]$). Problems 4 and 5 apply to P_1 and P_2. We prove the two problems for γ not necessarily differentiable.

For Problem 4, if γ does not intersect negative real axis, by picking δ small enough, $Re(\gamma(t)) > 2\delta > 0$ for all t. Therefore, if $|\gamma(t) - P_1(t)| < \delta$, P_1 does not intersect negative real axis either and we have $Ind(P_1) = 0$. Therefore, $Ind(\gamma) = 0$.

For Problem 5, suppose γ_1 and γ_2 satisfy $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$. Take trigonometric polynomials P_1, P_2 uniformly converge to γ_1 and γ_2 respectively. We then have

$$|1 - \frac{P_2}{P_1}| \leq |1 - \frac{\gamma_2}{\gamma_1}| + |\frac{\gamma_2}{\gamma_1} - \frac{P_2}{P_1}| < 1 + |\frac{\gamma_2}{\gamma_1} - \frac{P_2}{P_1}|.$$

But since P_1, P_2 converge uniformly to γ_1, γ_2, there exist P_1, P_2 so that $|1 - \frac{P_2(t)}{P_1(t)}| < 1$ for all t. We may then repeat Problem 5 for P_1 and P_2 to conclude that $Ind(P_1) = Ind(P_2)$ and therefore $Ind(\gamma_1) = Ind(\gamma_2)$