Name and Student ID's: ____________________

Homework 12, Advanced Calculus 1

1. Determine the Fourier series of the following functions on $[-\pi, \pi]$, in the form $\sum_{n\in\mathbb{Z}} c_n e^{inx}$:
   (a) $f(x) = 2 + 7\cos(3x) - 4\sin(2x.)$
   (b) $f(x) = x.$
   (c) $f(x) = (\pi - x)(\pi + x).$
2. Continue from Problem 1,
   (a) e^x.
   (b) $e^{|x|}$.
3. Let f be an even function of period 2π with $f(x) = \cos(2x)$ for $x \in [0, \frac{\pi}{2}]$ and $f(x) = -1$ for $x \in (\frac{\pi}{2}, \pi)$.
   (a) Complete the definition of f on its domain $[-\pi, \pi]$.
   (b) Find its Fourier series.
   (c) Compute the sum

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)(2k-1)(2k+3)}.$$

4. Find the Fourier series, in the form $a_0 + \sum a_n \cos(nx) + b_n \sin(nx)$, of the function

$$f(x) = (\pi - |x|)^2$$

   on $[-\pi, \pi]$.

   Use your result to prove the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

   (Hint: Check that $f(x)$ is an even function. What can you say about $f(x)\cos(nx)$ and $f(x)\sin(nx)$ and their integrals on $[-\pi, \pi]$?)
5. Given $f \sim \sum_n c_n e^{inx}$, f differentiable and $2\pi-$periodic, prove that $f' \sim \sum_n inc_n e^{inx}$.
6. Continue from Problem 5, suppose that both f and f' are equal to their Fourier series. Solve the differential equation

$$f'(x) + 2f(x - \pi) = \sin x.$$

7. Continue from Problem 5 and 6, suppose in addition that $f \in C^2$. Find all possible values of $a \in \mathbb{R}$ so that

$$f''(x) + af(x) = f(x + \pi),$$

for all $x \in \mathbb{R}$ and $f \neq 0$.

8. Rudin Chapter 8 Exercise 13

9. Rudin Chapter 8 Exercise 15

10. Rudin Chapter 8 Exercise 16

11. Rudin Chapter 8 Exercise 19

12. Rudin Chapter 8 Exercise 27