

(Hw 12)

1. (a) $f(x) = 2 + 7\cos(3x) - 4\sin(2x)$, $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (2 + 7\cos(3x) - 4\sin(2x)) \cdot e^{-inx} dx$

① $n=0$, then $\int_{-\pi}^{\pi} 2 + 7\cos(3x) - 4\sin(2x) dx = 4\pi \Rightarrow C_0 = 2$

② $n \neq 0$, then

$\int_{-\pi}^{\pi} 2 \cdot e^{-inx} dx = \frac{2}{-in} e^{-inx} \Big|_{-\pi}^{\pi} = 0$, $n = \pm 1, \pm 2, \dots \Rightarrow C_n = 0$, $n = \pm 1, \pm 2, \dots$

$\int_{-\pi}^{\pi} 7\cos(3x) \cdot e^{-inx} dx = \int_{-\pi}^{\pi} 7\cos(3x) (\cos(nx) - i\sin(nx)) dx = \begin{cases} 0, & n \neq \pm 3 \\ \int_{-\pi}^{\pi} 7\cos^2(3x) dx, & n = \pm 3 \end{cases}$

$= \begin{cases} 0, & n \neq \pm 3 \\ 7\pi, & n = \pm 3 \end{cases}$

$\Rightarrow C_n = \begin{cases} 0, & n \neq \pm 3 \\ \frac{7}{2}, & n = \pm 3 \end{cases}$

$\int_{-\pi}^{\pi} 4\sin(2x) \cdot e^{-inx} dx = \int_{-\pi}^{\pi} 4\sin(2x) \cdot (\cos(nx) - i\sin(nx)) dx = \begin{cases} 0, & n \neq \pm 2 \\ \int_{-\pi}^{\pi} -4i\sin^2(2x) dx = -4i\pi, & n = 2 \\ \int_{-\pi}^{\pi} 4i\sin^2(2x) dx = 4i\pi, & n = -2 \end{cases}$

$\Rightarrow C_n = \begin{cases} 0, & n \neq \pm 2 \\ -2i, & n = 2 \\ 2i, & n = -2 \end{cases}$

By ①, ②, then

$f(x) \sim 2 + (-2i)e^{-2ix} + 2i \cdot e^{2ix} + \frac{7}{2}e^{-3ix} + \frac{7}{2}e^{3ix}$

✘

$$(b) f(x) = x, \quad C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \cdot e^{-inx} dx$$

$$\textcircled{1} n=0, \text{ then } \int_{-\pi}^{\pi} x dx = \frac{1}{2} x^2 \Big|_{-\pi}^{\pi} = 0 \Rightarrow C_0 = 0$$

$\textcircled{2} n \neq 0$, then

$$\int_{-\pi}^{\pi} x \cdot e^{-inx} dx = \frac{-x}{in} \cdot e^{-inx} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{1}{in} \cdot e^{-inx} dx = \frac{-2\pi}{in} \cdot (-1)^n$$

$$u = x, \quad dv = e^{-inx} dx$$

$$du = dx, \quad v = \frac{1}{in} e^{-inx}$$

$$\Rightarrow C_n = \frac{(-1)^{n+1}}{in}, \quad n = \pm 1, \pm 2, \dots$$

By $\textcircled{1}, \textcircled{2}$, then

$$f(x) \sim \sum_{n \neq 0} \frac{(-1)^{n+1}}{in} \cdot e^{inx}$$

~~*~~

$$(c) f(x) = (\pi-x)(\pi+x) = \pi^2 - x^2, \quad C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi^2 - x^2) \cdot e^{-inx} dx$$

$$\textcircled{1} n=0, \text{ then } \int_{-\pi}^{\pi} (\pi^2 - x^2) dx = \frac{4}{3} \pi^3 \Rightarrow C_0 = \frac{2}{3} \pi^2$$

$\textcircled{2} n \neq 0$, then

$$\int_{-\pi}^{\pi} (\pi^2 - x^2) \cdot e^{-inx} dx = \int_{-\pi}^{\pi} \pi^2 \cdot e^{-inx} dx - \int_{-\pi}^{\pi} x^2 \cdot e^{-inx} dx = - \int_{-\pi}^{\pi} x^2 \cdot e^{-inx} dx$$

$$u = x^2, \quad dv = e^{-inx} dx$$

$$= \frac{x^2}{in} \cdot e^{-inx} \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2}{in} \cdot x \cdot e^{-inx} dx = \frac{-2}{in} \int_{-\pi}^{\pi} x \cdot e^{-inx} dx$$

$$du = 2x dx$$

$$v = \frac{1}{in} \cdot e^{-inx}$$

$$= \frac{-2}{in} \cdot \frac{-2\pi}{in} \cdot (-1)^n$$

$$\Rightarrow C_n = \frac{2}{n^2} \cdot (-1)^{n+1}, \quad n = \pm 1, \pm 2, \pm 3, \dots$$

By $\textcircled{1}, \textcircled{2}$, then

$$f(x) \sim \frac{2}{3} \pi^2 + \sum_{n \neq 0} (-1)^{n+1} \cdot \frac{2}{n^2} \cdot e^{inx}$$

~~*~~

2, (a)
 $f(x) = e^x$, $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x \cdot e^{-inx} dx$

① $n=0$, then $\int_{-\pi}^{\pi} e^x dx = e^{\pi} - e^{-\pi} \Rightarrow C_0 = \frac{e^{\pi} - e^{-\pi}}{2\pi}$

② $n \neq 0$, then $\int_{-\pi}^{\pi} e^x \cdot e^{-inx} dx = \int_{-\pi}^{\pi} e^{x(1-in)} dx = \frac{1}{1-in} \cdot e^{x(1-in)} \Big|_{-\pi}^{\pi}$
 $= \frac{1}{1-in} \cdot e^{\pi} \cdot \cos(n\pi) - \frac{1}{1-in} \cdot e^{-\pi} \cdot \cos(n\pi)$
 $\Rightarrow C_n = \frac{e^{\pi} - e^{-\pi}}{2\pi} \cdot \frac{(-1)^n}{1-in}$, $n = \pm 1, \pm 2, \dots$
 $= \frac{(-1)^n}{1-in} \cdot (e^{\pi} - e^{-\pi})$

By ①, ②, then

$f(x) \sim \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n \neq 0} \frac{(-1)^n}{1-in} \cdot \frac{e^{\pi} - e^{-\pi}}{2\pi} \cdot e^{inx}$ *

(b)
 $f(x) = e^{|x|}$, $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{|x|} \cdot e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^0 e^{-x} \cdot e^{-inx} dx + \frac{1}{2\pi} \int_0^{\pi} e^x \cdot e^{-inx} dx$

① $n=0$, then $\int_{-\pi}^{\pi} e^{|x|} dx = \int_{-\pi}^0 e^{-x} dx + \int_0^{\pi} e^x dx = e^{-\pi} - 1 + e^{\pi} - 1 = 2(e^{\pi} - 1) \Rightarrow C_0 = \frac{e^{\pi} - 1}{\pi}$

② $n \neq 0$, then $\int_{-\pi}^0 e^{-x} \cdot e^{-inx} dx = \int_{-\pi}^0 e^{x(-1-in)} dx = \frac{1}{-1-in} e^{(-1-in)x} \Big|_{-\pi}^0$
 $= \frac{1}{-1-in} + \frac{1}{1-in} \cdot e^{(1-in)\pi}$

$\int_0^{\pi} e^x \cdot e^{-inx} dx = \int_0^{\pi} e^{x(1-in)} dx = \frac{1}{1-in} \cdot e^{x(1-in)} \Big|_0^{\pi}$
 $= \frac{1}{1-in} \cdot e^{\pi(1-in)} - \frac{1}{1-in}$

$\Rightarrow C_n = \frac{1}{2\pi} \cdot \left[\frac{1}{-1-in} + \frac{1}{1-in} e^{(1-in)\pi} + \frac{1}{1-in} \cdot e^{\pi(1-in)} - \frac{1}{1-in} \right]$
 $C_n = \frac{1}{2\pi} \left[\frac{-2}{1+n^2} + \frac{2 \cdot e^{\pi} \cdot (-1)^n}{1+n^2} \right] = \frac{1}{2\pi} \cdot \frac{2(e^{\pi} \cdot (-1)^n - 1)}{1+n^2} = \frac{1}{\pi} \cdot \frac{e^{\pi} \cdot (-1)^n - 1}{1+n^2}$

By ①, ②, then $f(x) \sim \frac{e^{\pi} - 1}{\pi} + \sum_{n \neq 0} \frac{1}{\pi} \cdot \frac{e^{\pi} \cdot (-1)^n - 1}{1+n^2} \cdot e^{inx}$ *

3. f = even function, 2π -periodic, $f(x) = \begin{cases} \cos(2x), & 0 \leq x \leq \frac{\pi}{2}, \\ -1, & \frac{\pi}{2} \leq x < \pi. \end{cases}$

(a)

Since f is even, that is, $f(x) = f(-x)$, $\forall x \in [0, \pi)$, then we define

$$f(x) = \begin{cases} \cos(2x), & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ -1, & \frac{\pi}{2} \leq x < \pi \text{ or } -\pi < x \leq -\frac{\pi}{2}. \end{cases}$$

Now, $\lim_{x \rightarrow \pi^-} f(x) = \lim_{x \rightarrow \pi^-} (-1) = -1$, then we have $\lim_{x \rightarrow \pi^+} f(x) = \lim_{x \rightarrow \pi^+} f(x - 2\pi) = \lim_{x \rightarrow -\pi^+} f(x) = \lim_{x \rightarrow -\pi^+} (-1) = -1$.

So $\lim_{x \rightarrow \pi} f(x) = -1$. Define $f(\pi) = -1$. Then $f(-\pi) = -1$. Therefore $f(x)$ is continuous on \mathbb{R} .

(b)

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot e^{-inx} dx \text{ and } \int_{-\pi}^{-\frac{\pi}{2}} -e^{-inx} dx + \int_{\frac{\pi}{2}}^{\pi} \cos(2x) \cdot e^{-inx} dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -1 \cdot e^{-inx} dx \quad (*)$$

① $n=0$, then $(*) = \frac{-\pi}{2} + 0 + \frac{\pi}{2} = -\pi \Rightarrow C_0 = \frac{-1}{2}$.

② $n \neq 0$, then $(*) = \int_{-\pi}^{-\frac{\pi}{2}} -e^{-inx} dx + \int_{\frac{\pi}{2}}^{\pi} \cos(2x) \cdot e^{-inx} dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -e^{-inx} dx$

$$= \frac{ie^{\frac{i n \pi}{2}} (-1 + e^{\frac{i n \pi}{2}})}{n} - \frac{2n \sin(\frac{n \pi}{2})}{n^2 - 4} + \frac{ie^{-i n \pi} (-1 + e^{\frac{i n \pi}{2}})}{n}$$

$$= \frac{-ie^{\frac{i n \pi}{2}} + ie^{-\frac{i n \pi}{2}}}{n} - \frac{2n}{n^2 - 4} \cdot \sin(\frac{n \pi}{2})$$

$$\equiv \frac{-ie^{\frac{i n \pi}{2}} + ie^{-\frac{i n \pi}{2}}}{n} = \frac{h}{h^2 - 4} \cdot \frac{-ie^{\frac{i n \pi}{2}} + ie^{-\frac{i n \pi}{2}}}{h} = \frac{-4}{h^2 - 4} e$$

$n \neq 0$, then

$$C_n = \dots$$

$$(*) = \frac{-4}{(n^2-4)n} \cdot \left(-ie^{\frac{i n \pi}{2}} + ie^{-\frac{i n \pi}{2}} \right) = \frac{-8}{n(n+2)(n-2)} \cdot \sin\left(\frac{n\pi}{2}\right)$$

Let $n=2k-1$, then

$$(*) = \frac{-8}{(2k+1)(2k+3)(2k-1)} \cdot \sin\left(\frac{(2k-1)\pi}{2}\right), \quad k=1, 2, 3, \dots$$

$$= \frac{-8}{(2k-1)(2k+1)(2k+3)} \sin\left(k\pi - \frac{\pi}{2}\right), \quad k=1, 2, 3, \dots$$

$$= \frac{-8}{(2k-1)(2k+1)(2k+3)} \cdot (-1)^{k+1}, \quad k=1, 2, 3, \dots$$

$$\Rightarrow C_k = \frac{-4}{\pi(2k-1)(2k+1)(2k+3)} \cdot (-1)^k, \quad k=1, 2, 3, \dots$$

Then, we have

$$f(x) \sim \frac{-1}{2} + \sum_{k=1}^{\infty} \frac{-4}{\pi(2k-1)(2k+1)(2k+3)} \cdot (-1)^k \cdot e^{i(2k-1)x}$$

$$+ \sum_{k=1}^{\infty} \frac{-4}{\pi(2k-1)(2k+1)(2k+3)} \cdot (-1)^k \cdot e^{-i(2k-1)x}$$

(c) As $x \rightarrow 0$, then $f(0) = 1 = \frac{-1}{2} + \sum_{k=1}^{\infty} \frac{-4}{\pi} \frac{(-1)^k}{(2k-1)(2k+1)(2k+3)} \cdot x^2$

$$M \approx \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)(2k+1)(2k+3)}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-1)(2k+1)(2k+3)} = \frac{-3\pi}{16}$$

$$\frac{3}{16}$$

$$f(x) = (\pi - |x|)^2 \text{ on } [-\pi, \pi]$$

Sol

$$f(-x) = (\pi - |-x|)^2 = (\pi - |x|)^2 = f(x) \Rightarrow f(x) \text{ is even function.}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

$n=0, 1, 2, 3, \dots$ $n=1, 2, 3, \dots$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 dx = \frac{1}{2\pi} \left[\int_{-\pi}^0 (\pi + x)^2 dx + \int_0^{\pi} (\pi - x)^2 dx \right] = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 \cos(nx) dx = \frac{1}{\pi} \int_{-\pi}^0 (\pi + x)^2 \cos(nx) dx + \frac{1}{\pi} \int_0^{\pi} (\pi - x)^2 \cos(nx) dx$$

(even function)

$$= \frac{1}{\pi} \cdot \frac{2\pi h - 2\sin(n\pi)}{h^3} + \frac{1}{\pi} \cdot \frac{2\pi h - 2\sin(n\pi)}{h^3}$$

$$= \frac{4\pi h - 4\sin(n\pi)}{\pi h^3} = \frac{4}{h^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 \sin(nx) dx = \frac{1}{\pi} \int_{-\pi}^0 (\pi + x)^2 \sin(nx) dx + \frac{1}{\pi} \int_0^{\pi} (\pi - x)^2 \sin(nx) dx$$

(odd function)

$$= \frac{1}{\pi} \cdot \frac{-\pi^2 h^2 - 2\cos(n\pi) + 2}{h^3} + \frac{1}{\pi} \cdot \frac{\pi^2 h^2 + 2\cos(n\pi) - 2}{h^3}$$

$$= 0$$

So $f(x) \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4 \cos(nx)}{h^2}$ and $f(0) = \pi^2$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{h^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \times \frac{2\pi^2}{3} = \frac{\pi^2}{6}$$

*

$$\textcircled{4} \quad \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^4 dx = \frac{1}{2\pi} \cdot \frac{2\pi^5}{5} = \frac{\pi^4}{5}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|)^4 \cos(nx) dx = \frac{1}{\pi} \int_{-\pi}^0 (\pi + x)^4 \cos(nx) dx + \frac{1}{\pi} \int_0^{\pi} (\pi - x)^4 \cos(nx) dx \\ &= \frac{4(n^3\pi^3 - 6\pi n + 6\sin(n\pi))}{\pi n^5} + \frac{4(n^3\pi^3 - 6\pi n + 6\sin(n\pi))}{\pi n^5} \\ &= \frac{8(n^3\pi^3 - 6\pi n)}{\pi n^5} = \frac{8n^2\pi^2 - 48}{n^4} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - |x|)^4 \sin(nx) dx = \frac{1}{\pi} \int_{-\pi}^0 (\pi + x)^4 \sin(nx) dx + \frac{1}{\pi} \int_0^{\pi} (\pi - x)^4 \sin(nx) dx \\ &= \frac{-\pi^4 n^4 + 12\pi^2 n^2 + 24\cos(n\pi) - 24}{\pi n^5} + \frac{\pi^4 n^4 - 12\pi^2 n^2 - 24\cos(n\pi) + 24}{\pi n^5} \\ &= 0 \end{aligned}$$

$$S_0(f(x))^2 \sim \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8n^2\pi^2 - 48}{n^4} \cdot \cos(nx) \quad \text{and} \quad (f(0))^2 = \pi^4$$

$$\Rightarrow \pi^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{8\pi^2}{n^2} - \sum_{n=1}^{\infty} \frac{48}{n^4}$$

$$\Rightarrow \frac{4\pi^4}{5} = 8\pi^2 \cdot \frac{\pi^2}{6} - 48 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \frac{4\pi^4}{5} = \frac{4\pi^4}{3} - 48 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{48} \times \frac{8\pi^4}{15} = \frac{\pi^4}{90}$$

*

Ex. Given $f \sim \sum_n c_n \cdot e^{inx}$ and f is differentiable, prove that $f' \sim \sum_n in c_n e^{inx}$.
 2π -periodic

(pf)

Since $f \sim \sum_n c_n \cdot e^{inx}$, then $c_n = \int_{-\pi}^{\pi} f(x) \cdot e^{-inx} dx$.

Claim: $f' \sim \sum_n in c_n \cdot e^{inx}$, that is, $\int_{-\pi}^{\pi} f'(x) \cdot e^{-inx} dx = in c_n$, $\forall n \in \mathbb{N}$

$$\therefore \int_{-\pi}^{\pi} f'(x) \cdot e^{-inx} dx = \underbrace{f(x) \cdot e^{-inx}}_{\substack{\text{red wavy line} \\ \nearrow 0}} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} f(x) \cdot in \cdot e^{-inx} dx, \quad \forall n \in \mathbb{N}$$

$$u = e^{-inx} \quad dv = f'(x) dx \quad \text{and} \quad \underline{f(\pi) = f(-\pi)}$$

$$du = -in e^{-inx} dx \quad v = f(x)$$

$$\Rightarrow \int_{-\pi}^{\pi} f'(x) \cdot e^{-inx} dx = in \int_{-\pi}^{\pi} f(x) \cdot e^{-inx} dx = in c_n, \quad \forall n \in \mathbb{N}.$$

$$\text{So } f'(x) \sim \sum_n in c_n \cdot e^{inx}.$$



6. By problem 5, f and f' are equal to their Fourier series and f is 2π -periodic.

Solve the differential equation $f'(x) + 2f(x-\pi) = \sin x$.

<sol>

$$\textcircled{1} \text{ Now, } f(x) = \sum_n c_n \cdot e^{inx} \text{ and } f'(x) = \sum_n in c_n \cdot e^{inx}, \forall x \in \mathbb{R}.$$

$$f(x-\pi) = \sum_n c_n \cdot e^{in(x-\pi)} = \sum_n c_n \cdot e^{inx} \cdot e^{-in\pi}$$

$$\text{Now, } e^{-in\pi} = \begin{cases} (-1)^n, & n=0, 1, 2, \dots \\ (-1)^{-n}, & n=-1, -2, \dots \end{cases} \Rightarrow f(x-\pi) = \sum_{n=0}^{\infty} c_n \cdot (-1)^n \cdot e^{inx} + \sum_{n=-1}^{-\infty} c_n \cdot (-1)^{-n} \cdot e^{inx}$$

$$\textcircled{2} \quad f'(x) + 2f(x-\pi) = \sin x \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{-i}{2} e^{ix} + \frac{i}{2} e^{-ix}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} in c_n e^{inx} + \sum_{n=0}^{\infty} 2c_n \cdot (-1)^n \cdot e^{inx} + \sum_{n=-1}^{-\infty} 2c_n \cdot (-1)^{-n} \cdot e^{inx} = \frac{-i}{2} e^{ix} + \frac{i}{2} e^{-ix}$$

$$\Rightarrow \sum_{n=0}^{\infty} (in c_n + 2c_n \cdot (-1)^n) \cdot e^{inx} + \sum_{n=-1}^{-\infty} (in c_n + 2c_n \cdot (-1)^{-n}) \cdot e^{inx} = \frac{-i}{2} e^{ix} + \frac{i}{2} e^{-ix}$$

$$\Rightarrow c_0 = 0, \quad c_n = 0, \quad \forall n \neq \pm 1, \quad \text{and} \quad \begin{cases} ic_1 - 2c_1 = \frac{-i}{2} \\ -ic_{-1} - 2c_{-1} = \frac{i}{2} \end{cases}$$

$$\Rightarrow c_0 = 0, \quad c_n = 0, \quad \forall n \neq \pm 1, \quad c_1 = \frac{i}{4-2i}, \quad \text{and} \quad c_{-1} = \frac{-i}{4+2i}$$

$$\text{So } f(x) = \frac{i}{4-2i} e^{ix} - \frac{i}{4+2i} e^{-ix}$$

✘



7. By problem 5, and problem 6, $f \in C^2$, f is 2π -periodic. Find all possible values of $a \in \mathbb{R}$ such that $f''(x) + a f(x) = f(x + \pi)$ for all $x \in \mathbb{R}$.

<sol>

① Now, $f(x) = \sum_n c_n e^{inx}$, $f'(x) = \sum_n in \cdot c_n \cdot e^{inx}$, $\forall x \in \mathbb{R}$.

$$f(x+\pi) = \sum_n c_n \cdot e^{in(x+\pi)} = \sum_n c_n \cdot e^{inx} \cdot e^{in\pi}, \quad e^{in\pi} = \begin{cases} (-1)^n, & n=0,1,2,\dots \\ (-1)^{-n}, & n=-1,-2,\dots \end{cases}$$

$$\Rightarrow f(x+\pi) = \sum_{n=0}^{\infty} c_n \cdot (-1)^n \cdot e^{inx} + \sum_{n=-1}^{-\infty} c_n \cdot (-1)^{-n} \cdot e^{inx}$$

Now, $f''(x) = \sum_n -n^2 \cdot c_n \cdot e^{inx}$, $\forall x \in \mathbb{R}$ ($\because f \in C^2 \Rightarrow f''$ is continuous)

② $f''(x) + a f(x) = f(x+\pi)$

$$\Rightarrow \sum_n -n^2 c_n \cdot e^{inx} + \sum_n a c_n \cdot e^{inx} = \sum_{n=0}^{\infty} c_n \cdot (-1)^n \cdot e^{inx} + \sum_{n=-1}^{-\infty} c_n \cdot (-1)^{-n} \cdot e^{inx}$$

$$\Rightarrow \sum_{n=0}^{\infty} (-n^2 c_n + a c_n - c_n \cdot (-1)^n) \cdot e^{inx} + \sum_{n=-1}^{-\infty} (-n^2 c_n + a c_n - c_n \cdot (-1)^{-n}) \cdot e^{inx} = 0$$

\Rightarrow Since $\{e^{inx}\}_{n=-\infty}^{n=\infty}$ is an orthonormal basis, then we have

$$(a - n^2 - (-1)^n) \cdot c_n = 0, \quad n=0,1,2,3,\dots \quad \text{and} \quad (a - n^2 - (-1)^{-n}) \cdot c_n = 0, \quad n=-1,-2,-3,\dots$$

$$\Rightarrow \left\{ \begin{array}{l} (a-1) c_0 = 0 \\ (a-0) c_1 = 0 \\ (a-5) c_2 = 0 \\ (a-8) c_3 = 0 \\ \vdots \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} (a-0) c_{-1} = 0 \\ (a-5) c_{-2} = 0 \\ (a-8) c_{-3} = 0 \\ \vdots \end{array} \right.$$

③ Observe that

$$(1) a \neq \begin{cases} n^2 - (-1)^n, & n=0, 1, 2, \dots \\ n^2 - (-1)^{-n}, & n=-1, -2, \dots \end{cases}$$

Then $c_n = 0 \quad \forall n \in \mathbb{Z} \Rightarrow f(x) \equiv 0 \quad \forall x \in \mathbb{R}$.

(2)

$a = n_0^2 - (-1)^{n_0}$ for some $n_0 > 0$. Then $a = (-n_0)^2 - (-1)^{-(-n_0)}$ for some $-n_0 < 0$.

$$\Rightarrow c_n = 0 \quad \forall n \in \mathbb{Z} \setminus \{n_0, -n_0\}$$

$$\Rightarrow f(x) = c_{n_0} \cdot e^{in_0x} + c_{-n_0} \cdot e^{-in_0x} \quad \text{for some } n_0 > 0.$$

✘