Name and Student ID's: ____________________

Homework 1, Advanced Calculus 1

1. Prove Proposition 1.15 of Rudin (page 7).

> **Solution:** Since $x \neq 0$, $1/x$ exists and we may multiply it to the equations in part (a) and (b). Using associativity of multiplication (M2), the results follow. For part (c), we simply note that $xy = 1 = x(1/x)$, and by part (b), $y = 1/x$. Part (d) follows from the equation $1 = (1/x)(1/(1/x)) = (1/x)x$ and part (a).

2. In class we have defined the rational number $\mathbb{Q}$ as the set of equivalence classes:

$$\mathbb{Q} := (\mathbb{Z} \times \mathbb{Z}\backslash\{0\})/\sim,$$

where $(a,b) \sim (c,d) \Leftrightarrow ad = bc$. Define field operations on $\mathbb{Q}$ by

- $[(a,b)] + [(c,d)] = [(ad+bc, bd)]$.
- $[(a,b)] \cdot [(c,d)] = [(ac, bd)]$.

(a) Show that the operations above are well defined. That is, different choices of representative from the classes give the same equivalence classes on the right hand side.

(b) Prove that $\mathbb{Q}$ with these operations is a field. You do not need to show associativity and commutativity in both operations.

> **Solution:**
>
> (a) Take equivalent pairs $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ in $\mathbb{Z} \times \mathbb{Z}\backslash\{0\}$. By definition we have $ab' = a'b$ and $cd' = c'd$. We then have $(ad+bc)(b'd') = adb'd' + bcb'd' = a'bdd' + bc'c'd = (a'd' + b'c')bd$. Therefore $(ad+bc, bd) \sim (a'd'+b'c', b'd')$ and therefore $[(ad+bc, bd)] = [(a'd'+b'c', b'd')]$ so addition is independent of choice of representatives and well defined. The well-definedness for multiplication is similar.
>
> (b) The operations above immediately imply that $(\mathbb{Q}, +, \cdot)$ is a field with $0 = [(0,1)]$ and $1 = [(1,1)]$. The additive inverse of every $[(a,b)]$ is given by $[(-a,b)]$ and the multiplicative inverse of $[(a,b)]$ with $a \neq 0$ is simply $[(b,a)]$. It is elementary to check all field axioms.

3. Rudin Exercise 6 ab.

> **Solution:**
>
> (a) By Theorem 1.21, the equality we want to prove is equivalent to
>
> $$b^m = ((b^p)^{1/q})^n. \tag{1}$$
>
> But by Corollary after Theorem 1.21, $(b^p)^{1/q} = (b^{1/q})^p$, and $((b^p)^{1/q})^n = (b^{1/q})^{pn}$. But from $m/n = p/q$, we have $pn = qm$. The right hand side of (1) is then $((b^p)^{1/q})^n = (b^{1/q})^{qm}$, which is just b^m.

(b) Let $r = m/n$ and $s = p/q$, we have

$$r + s = \frac{mq + np}{nq},$$

and so

$$\begin{aligned} b^{r+s} &= (b^{mq+np})^{\frac{1}{nq}} \\ &\overset{\text{Cor.}}{=} (b^{mq})^{\frac{1}{nq}}(b^{np})^{\frac{1}{nq}} \\ &\overset{\text{Thm.,Cor.}}{=} b^{\frac{m}{n}} b^{\frac{p}{q}} \\ &= b^r b^s. \end{aligned}$$

4. Rudin Exercise 6 cd.

Solution:

(a) For all $x \in \mathbb{R}$, the set

$$B(x) := \{b^t \mid t \leq x, t \in \mathbb{Q}\}$$

is clearly nonempty. We check that it is bounded above (and therefore sup exists and hence the definition b^x is a valid statement. It is natural to expect that $B(x)$ is bounded above by b^s, where s is any rational number greater than x, and therefore greater than all rational numbers $\leq x$. For this we must check that the quantity b^r is monotonic in r for $r \in \mathbb{Q}$.

Take $r, s \in \mathbb{Q}$ with $r < s$. We have from 6b that $b^s - b^r = b^r(b^{s-r} - 1)$ with $s - r = \frac{n}{m} > 0$. By 6a, we may assume that $m, n > 0$. We have that $(b^{s-r})^m = b^n > 1$ since $b > 1$ and $n > 0$. Therefore $b^{s-r} > 1$ (otherwise its mth power would be ≥ 1.) It then follows that $b^s - b^r > 0$ and we are done.

We must also check that the definition of b^x coincides with the one given in 6a when $x \in \mathbb{Q}$. But it follows easily from the monotonicity argument. If $x \in \mathbb{Q}$, $b^x \in B(x)$ and $b^x \geq b^t \ \forall$ rational $t \leq x$. So $B(x)$ contains an upper bound of itself and therefore must be the least upper bound (anything smaller than b^x is not an upper bound of $B(x)$ since it is smaller than $b^x \in B(x)$.)

(b) We first show that $b^x b^y$ is an upper bound of the set $B(x + y)$. Take $b^t \in B(x + y)$, where t is rational and $\leq x + y$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists rational number $t_1 \in [t - y, x]$. Therefore $t_1 \leq x$ and $t - t_1 \leq y$. Take another rational number $t_2 \in [t - t_1, y]$. We then have $t_2 \leq y$ and $t \leq t_1 + t_2 \leq x + y$. By monotonicity proved in 6c and power rule 6d, we have

$$b^t \leq b^{t_1+t_2} = b^{t_1} b^{t_2}.$$

But by our choices of t_1 and t_2, we know that $b^{t_1} \in B(x)$ and $b^{t_2} \in B(y)$, and therefore are no greater than the corresponding supremums b^x and b^y, respectively.

To show that $b^x b^y$ is the least upper bound, we pick any $\epsilon > 0$. It is possible to pick $\epsilon_1, \epsilon_2 > 0$ so that

$$b^x b^y - \epsilon \leq (b^x - \epsilon_1)(b^y - \epsilon_2)$$

since the term $\epsilon_1 b^y + \epsilon_2 b^x - \epsilon_1\epsilon_2 \to 0$ as $\epsilon_1, \epsilon_2 \to 0$. Then, by definition of supremum, there exists rational numbers t_1, t_2 no greater than x, y, respectively, so that $(b^x - \epsilon_1) \le b^{t_1}$ and $(b^y - \epsilon_2) \le b^{t_2}$. Therefore, $b^x b^y - \epsilon \le b^{t_1} b^{t_2} = b^{t_1+t_2} \in B(x+y)$.

5. Rudin Exercise 7 abcd.

Solution:

(a) The estimate follows simply from

$$b^n - 1 = (b-1)(b^{n-1} + b^{n-2} + \cdots + 1) \ge n(b-1)$$

since $b \ge 1$ and each of the n terms in the last parenthesis is no less than 1.

(b) Since $b > 1$, its n^{th} root $b^{\frac{1}{n}} > 1$ and we simply replace b by $b^{\frac{1}{n}}$ in part (a).

(c) The result follows simply by rearranging the inequality in part (b) and replace 1 by t.

(d) Since $y > b^w$, we have $t = yb^{-w} > 1$. By Archimedean property there exists $n \in \mathbb{N}$ that is greater than $\frac{b-1}{yb^{-w}-1}$. For such n and t, part (c) implies that $b^{\frac{1}{n}} < t = yb^{-w}$ and therefore $y > b^{w+\frac{1}{n}}$.

6. Rudin Exercise 7efg.

Solution:

(a) Since $y < b^w$, we have $t = b^w y^{-1} > 1$. Like in 6d, we pick $n \in \mathbb{N}$ greater than $\frac{b-1}{y^{-1}b^w - 1}$ to arrive at the conclusion $b^{\frac{1}{n}} < y^{-1} b^w$, or $y < b^{w - \frac{1}{n}}$.

(b) Given $A = \{w \in \mathbb{R} \mid b^w < y\}$. The set is nonempty since there exists some $n \in \mathbb{Z}$ so that $b < y^n$ and therefore $b^{\frac{1}{n}} < y$ or $\frac{1}{n} \in A$. It is clearly bounded above since there exists $n \in \mathbb{N}$ so that $b^n > y$ (recall $b > 1$) and therefore all $w \in A$ must be no greater than n. Therefore $x = \sup A$ exist.

The next two problems deal with the *decimal expansion* of real numbers.

7. Given a real number $x > 0$,
   (a) prove that there is a *largest* integer $n_0 \le x$. (Use Archimedean property)
   (b) Inductively, for each $k \in \mathbb{N}$, let n_k be the largest integer so that

$$n_k \le 10^k \left(x - n_0 - n_1 10^{-1} - \cdots - n_{k-1} 10^{-(k-1)} \right),$$

   or equivalently

$$A_k = \sum_{j=0}^{k} n_j 10^{-j} \le x.$$

   Show that $0 \le n_j \le 9$ for all $j > 1$.

(c) Prove that the sequence $E = \{A_k\}$ is monotonic and bounded above, and therefore $\lim_k A_k$ exists and is equal to $\sup E$.

Solution:

(a) By Archimedean property, the set $\{n \mid n > x\} \subset \mathbb{N}$ is nonempty and therefore has a minimum element n_0'. Then $n_0 := n_0' - 1$ is the desired integer.

(b) E is certainly bounded above element is no greater than x. Moreover, $A_k - A_{k-1} = n_k 10^{-k} \geq 0$ and therefore the sequence is monotonic and the limit is precisely the supremum.

8. (a) Prove that $x = \sup E$.

(b) Eliminating sequences $\{n_j\}$ mentioned above with the property that $n_j = 9$ for all j after a certain term (which is impossible from its construction anyway), prove that

$$\sum_{j=0}^{\infty} n_j 10^{-j} = \sum_{j=0}^{\infty} m_j 10^{-j} \Rightarrow n_j = m_j \ \ \forall j.$$

Solution:

(a) We show that $x = \lim_k A_k$. Indeed, the constructions of n_k imply that

$$0 \leq (x - A_k)10^k - n_k \leq 1,$$

since n_k is the largest integer $\leq (x - A_k)10^k$. Or

$$0 \leq x - A_k \leq \frac{1 + n_k}{10^k} \leq \frac{1}{10^{k-1}}$$

since $n_k \leq 9$. Letting $k \to \infty$, the result follows.

(b) Suppose the contrary, and let N be the first digit where $n_N \neq m_N$, or without loss of generality, $n_N > m_N$. Since the infinite series converge, we may subtract term-by-term

$$0 = (n_N - m_N)10^{-N} + \sum_{j=N+1}^{\infty} (n_j - m_j)10^{-j}.$$

Since we eliminate decimals with repeated 9's, the absolute value of the second term is estimated by

$$|\sum_{j=N+1}^{\infty} (n_j - m_j)10^{-j}| < \sum_{j=N+1}^{\infty} 9 \cdot 10^{-j} = 10^{-N}$$

and the first term is at least 10^{-N}. Therefore, their sum is strictly greater than 0, a contradiction.

We have shown that every positive real number may be uniquely expressed by an infinite sequence of integers $\{n_j\}$ with $0 \leq n_j \leq 9 \ \ \forall j > 1$ so that

$$x = \sum_{j=0}^{\infty} n_j 10^{-j}.$$

We usually denote it by

$$x = n_0.n_1 n_2 n_3 \cdots ,$$

and call it the *decimal expansion* of x. Note that we may replace 10 by any other positive integer $N > 1$ and the entire construction holds without any major modification (you may be familiar with the expansion with $N = 2$).