Name and Student ID's: ____

Homework 3, Advanced Calculus 1

For the first two problems, we consider the space for a real number $p \geq 1$:

$$H^p := \{f : \mathbb{R} \to \mathbb{R} \ \mathit{continuous} \mid \int_{\mathbb{R}} |f|^p dx < \infty\}.$$

1. Prove that H^p is closed under usual addition and scalar multiplication of functions. That is, for every $f, g \in H^p$ and $a \in \mathbb{R}$, we have $af, f + g \in H^p$.

2. On H^p, we define $d : H^p \times H^p \to \mathbb{R}$ by

$$d(f, g) := \left(\int_{\mathbb{R}} |f - g|^p\right)^{\frac{1}{p}}.$$

Prove that d satisfies the triangle inequality

$$d(f, g) \leq d(f, h) + d(h, g)$$

and therefore is a metric on H^p. (The other requirements are quite easy to check.)

(Hint: For this problem, it is helpful to use the *Hölder inequality*:

For all $p, q \in [1, \infty]$ so that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_{\mathbb{R}} |fg|dx \leq \left(\int_{\mathbb{R}} |f|^p\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g|^q\right)^{\frac{1}{q}}.$$

Start by usual triangle inequality $|F + G|^p \leq (|F| + |G|)|F + G|^{p-1}$.

3. Prove that for *any* metric space (X, d), finite sets are closed.

4. Prove that $\mathbb{R}^n$ with *discrete metric*, ie. the one given in Exercise 10 in Rudin, is a bounded set.

Two metrics d_1 and d_2 on a set X are said to be equivalent if there exist constants $C_1, C_2 > 0$ so that

$$C_1 d_2(x, y) \leq d_1(x, y) \leq C_2 d_2(x, y)$$

for all $x, y \in X$.

It is straightforward to show that the relation described above is an equivalence relation. (Problem 9)

5. If two metrics d_1 and d_2 are equivalent on X, prove that they induce the same topology. That is, a subset $E \subset X$ is open with respect to d_1 if and only if E is open with respect to d_2. Prove the same statement with "open" replaced by "closed"
6. Prove that on $\mathbb{R}$, the discrete metric (the one in Exercise 10 of Rudin) is *not* equivalent to the Euclidean metric $d_{EC}(x, y) = |x - y|$.
7. Prove that on $\mathbb{R}^n$, d_{EC} is equivalent to the *infinity metric*:

$$d_\infty(\mathbf{x}, \mathbf{y}) := max_{1 \leq j \leq n}\{|x_j - y_j|\}.$$

8. Prove that the metric d_5 in Exercise 11 of Rudin is *not* equivalent to d_∞.
9. Show that two metrics being equivalent is an equivalence relation.