

HW 5.

1. Rudin, ch 2, *18

Is there a nonempty perfect set in \mathbb{R}' which contains no rational number?

Answer: Yes

(pf) Let C be cantor set of $[0,1]$.

Let $x \in \mathbb{R}$ and $E = [x] + C$.

Since C is perfect, then E is perfect set.

Claim: $\exists z \in \mathbb{R}$ st. $(\{z\} + C) \cap \mathbb{Q} = \emptyset$.

Suppose not, then $\exists z \in \mathbb{R}$, $\forall z \in \mathbb{R}$, $(\{z\} + C) \cap \mathbb{Q} \neq \emptyset$.

Given $z \in \mathbb{R}$, then $\exists a \in \mathbb{Q}$ st. $a = z + c$ for some $c \in C$.

$$\Rightarrow z = a - c = \underbrace{-c}_{\in C} + \underbrace{a}_{\in \mathbb{Q}} = (-c) + (a - c) \in C + \mathbb{Q}$$

$$\Rightarrow \mathbb{R} \subseteq C + \mathbb{Q}. \quad (\text{利用對稱性})$$

Since $C + \mathbb{Q} \subseteq \mathbb{R}$, then $\mathbb{R} = C + \mathbb{Q}$.

Let $\mathbb{Q} = \{r_1, r_2, \dots\}$. Then $\mathbb{R} = \bigcup_{n=1}^{\infty} (\{r_n\} + C)$.

Since C is closed, then $\{r_n\} + C$ is also closed set $\forall n \in \mathbb{N}$.

By ch 2, Ex 30, then $\{r_{n_0}\} + C$ has an nonempty interior for some $n_0 \in \mathbb{N}$. -(*)

Since C has no interior point, then $\{r_{n_0}\} + C$ has no interior point $\forall n \in \mathbb{N}$.
(contradict to (*))

Thus, $\exists z \in \mathbb{R}$ st. $(\{z\} + C) \cap \mathbb{Q} = \emptyset$.



2. Rudin, ch2, * (9(c)(d))

(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$,
define B similarly with $>$ in place of $<$. Prove that A and B are separable.

\leftarrow pf

Let $A = \{q \in X \mid d(p, q) < \delta\}$ and $B = \{q \in X \mid d(p, q) > \delta\}$.

Since A and B are disjoint open sets, by Ex(9c6),

then A and B are separable. \blacksquare

(d) prove that every connected space with at least two points is uncountable.

\leftarrow pf

Let X be connected space with $x \neq y$, $x, y \in X$.

$\forall x \neq y$, then $d(x, y) > 0$.

Let $\delta = d(x, y) > 0$.

Given $0 < k < 1$. Let $A = \{z \in X \mid d(z, x) < \delta k\}$ and $B = \{z \in X \mid d(z, x) > \delta k\}$.

By (c), then A and B are separable.

Suppose $A \cup B = X$. Since A and B are separable, then X is not connected.
(contradiction)

So $A \cup B \neq X$, that is, $\exists y_k \in X$ with $y_k \notin A$, $y_k \notin B$, and $d(y_k, x) = \delta k$.
(y_k depend of k)

Define: $f: (0, 1) \rightarrow X$

$\underline{k} \mapsto \underline{y_k} \in X$ with $d(y_k, x) = \delta k$.

(by above discussion to choose)

Claim: f is injective.

Let $f(k) = f(l)$. Then $y_k = y_l$ with $d(y_k, x) = \delta k$ and $d(y_l, x) = \delta l$.

$\because y_k = y_l \Rightarrow d(y_k, x) = d(y_l, x) \Rightarrow \delta k = \delta l ; \delta > 0 \Rightarrow k = l$.

So f is injective.

Let $C = f((0,1))$, Then $f: (0,1) \rightarrow C$ is one-to-one correspondence. (bijective)

$\because (0,1)$ is uncountable, then C is also uncountable.

$\because C \subseteq X$, then X is also uncountable. ■

3. Rudin, ch2, #20

Are closures and interiors of connected sets always connected? (Look at subsets of \mathbb{R}^2)

(pf) Let E be connected set.

Claim: \bar{E} is connected set.

Suppose \bar{E} is not connected.

Then there are two sets G and H s.t. G and H are separable with $G \cup H = \bar{E}$.
 $\Rightarrow \bar{G} \cap H = \emptyset, G \cap \bar{H} = \emptyset, G \neq \emptyset, H \neq \emptyset.$

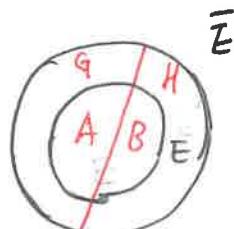
Let $A = E \cap G$ and $B = E \cap H$. Then $A \cup B = E$.

$$A \cap B = (\bar{E} \cap G) \cap (\bar{E} \cap H) \subseteq \bar{E} \cap \bar{G} \cap \bar{E} \cap \bar{H} \subseteq \bar{G} \cap \bar{H} = \emptyset \Rightarrow A \cap B = \emptyset.$$

$$A \cap \bar{B} = (\bar{E} \cap G) \cap (\bar{E} \cap \bar{H}) \subseteq \bar{E} \cap G \cap \bar{E} \cap \bar{H} \subseteq G \cap \bar{H} = \emptyset \Rightarrow A \cap \bar{B} = \emptyset.$$

Claim: $A \neq \emptyset$ and $B \neq \emptyset$.

Suppose $A = \emptyset$, then $B = E \cap H = \emptyset$.



Then $E \subseteq H$.

$\because \bar{E} = E \cup E' = G \cup H$, $E \subseteq H$, $G \neq \emptyset$, then $E' \cap G \neq \emptyset$.

$$\Rightarrow \exists z \in E' \cap G \Rightarrow z \in E' \text{ and } z \in G.$$

$\because E \subseteq H \Rightarrow \bar{E} \subseteq \bar{H}$. $\because z \in E' \Rightarrow z \in \bar{E} \Rightarrow z \in \bar{H}$.

$\therefore z \in \bar{H} \cap G$, but $\bar{H} \cap G = \emptyset$. (contradiction)

$\therefore A \neq \emptyset$. Similarly, $B \neq \emptyset$. Thus, A and B are separable.

Since $E = A \cup B$, then E is not connected. (contradiction)

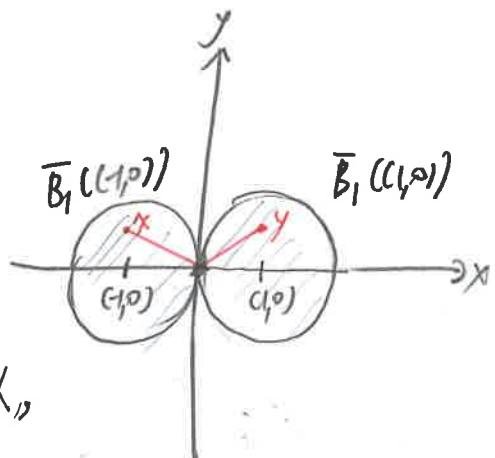
$\therefore \bar{E}$ is connected.

□

$\left\langle \text{sol} \right\rangle$

E° is not connected.

Ex: $E = A \cup B = \overline{B}_1((1,0)) \cup \overline{B}_1((-1,0))$



• Path-connected set:

$X \neq \emptyset$, $\forall x, y \in X$, if \exists continuous function $f: [0, 1] \rightarrow X$,

such that $f(0) = x$ and $f(1) = y$, then X is called "path-connected set."

• Path-connected set \Rightarrow connected set.

Clearly, E is path-connected set, then E is connected.

But $E^\circ = B_1((-1,0)) \cup B_1((1,0))$. Since $B_1((-1,0))$ and $B_1((1,0))$ are disjoint open sets,

then they are separable. So E° is not connected.

$\times \times$

4. Rudin, ch2, #27

$E \subseteq \mathbb{R}^k$, E is uncountable; let p be a condensation point of E , that is, every neighborhood of p contains uncountably many points of E .

Prove that P is perfect and that at most countably many points of E are not in P .
($E = P \cup (P^c \cap E)$, $P^c \cap E$ is at most countable.)

(pf)
Let $F = \{V_n\}_{n=1}^{\infty}$ be a countable base of \mathbb{R}^k .

$$E = E \cap \left(\bigcup_{n=1}^{\infty} V_n \right) = \bigcup_{n=1}^{\infty} (E \cap V_n) = \left[\bigcup_{E \cap V_n \text{ uncountable}} (E \cap V_n) \right] \cup \left[\bigcup_{E \cap V_n \text{ countable}} (E \cap V_n) \right]$$

Let $W = \bigcup_{E \cap V_n \text{ countable}} (E \cap V_n)$. Since the countable union of countable sets is also countable,
then W is countable.

Claim: $W^c = P$ and P is perfect set.

① Let $x \in W^c$ and O be any neighborhood of x .

Since $\{V_n\}_{n=1}^{\infty}$ is a countable base of \mathbb{R}^k , then $x \in V_n \subseteq O$ for some n .

$\because x \in W^c \Rightarrow x \notin W \Rightarrow V_n \cap E$ is uncountable

$\because V_n \cap E \subseteq O \cap E$ and $V_n \cap E$ is uncountable $\Rightarrow O \cap E$ is also uncountable $\Rightarrow x \in P$.

So $W^c \subseteq P$.

② If $x \in W$, then $x \in V_n$ for some V_n with $V_n \cap E$ is countable.

$\because V_n$ is open $\Rightarrow \exists r > 0$ s.t. $B_r(x) \subseteq V_n$ with $B_r(x) \cap E \subseteq V_n \cap E$ and $V_n \cap E$ is countable
(open neighborhood)

$\Rightarrow B_r(x) \cap E$ is also countable $\Rightarrow x \notin P \Rightarrow x \in P^c$. So $W \subseteq P^c$, that is, $P \subseteq W^c$.

Thus, $P = W^c$.

③ Since W is open, then $P = W^c$ is closed.

Claim: $P \subseteq P'$.

Let $x \in P$. and O be any neighborhood of x .

Then $O \cap E$ is uncountable.

Suppose $(O \setminus \{x\}) \cap P = \emptyset$. Then $O \setminus \{x\} \subseteq P^c$.

$\forall y \in O \setminus \{x\}$, since $\{V_n\}$ is countable base, then $\exists V_n$ s.t. $y \in V_n \subseteq O$.

$\because y \in O \setminus \{x\} \Rightarrow y \in P^c \Rightarrow y \notin P \Rightarrow V_n \cap E$ is countable $\Rightarrow y \in W$.

So $O \setminus \{x\} \subseteq W$

$\Rightarrow O \cap E = ((O \setminus \{x\}) \cup \{x\}) \cap E \subseteq (W \cup \{x\}) \cap E = (W \cap E) \cup (\{x\} \cap E) \subseteq \underbrace{(W \cap E)}_{(\text{countable})} \cup \{x\}$.

$\Rightarrow O \cap E$ is countable (contradiction)

Thus $(O \setminus \{x\}) \cap P \neq \emptyset$ for any neighborhood O of x . $\Rightarrow x \in P'$. So P is perfect set.



Rudin, ch2, *19 (a), (b),

(a) If A and B are disjoint closed sets in some metric space X , prove that they are separable.

(pf) Since A and B are closed, then $\bar{A} = A$ and $\bar{B} = B$.

$$\text{If } \bar{A} \cap B = A \cap B = \emptyset \text{ and } A \cap \bar{B} = A \cap B = \emptyset$$

$\therefore A$ and B are separable. □

(b)

If A and B are disjoint open sets in some metric space X , prove that they are separable.

(pf)

Since $A \cap B = \emptyset$, B is open, then $X \setminus B$ is closed set containing A .

By theorem 2.27, then $\bar{A} \subseteq (X \setminus B)$, that is, $\bar{A} \cap B = \emptyset$.

Since $A \cap B = \emptyset$, A is open, then $X \setminus A$ is closed set containing B .

By theorem 2.27, then $\bar{B} \subseteq (X \setminus A)$, that is, $\bar{B} \cap A = \emptyset$.

So A and B are separable. □

9. Rudin, ch2, ex2.

Let A and B be separated subsets of some \mathbb{R}^k , $a \in A$, $b \in B$, and define $p(t) = (1-t)a + tb$ for $t \in \mathbb{R}$. Put $A_0 = p^{-1}(A)$, $B_0 = p^{-1}(B)$.

(a) Prove that A_0 and B_0 are separated subset of \mathbb{R} .

(pf) ① Claim: $A_0 \cap B_0 = \emptyset$.

Suppose not, $\exists z \in A_0 \cap B_0$. Then $z \in A_0$ and $z \in B_0$.

$\Rightarrow p(z) \in A$ and $p(z) \in B \Rightarrow A \cap B \neq \emptyset$. (contradiction)

So $A_0 \cap B_0 = \emptyset$.

② Claim: $\overline{A_0} \cap B_0 = \emptyset$.

Suppose not, $\exists x \in \overline{A_0} \cap B_0$. Then $x \in \overline{A_0}$ and $x \in B_0$.

Since $A_0 \cap B_0 = \emptyset$, then $x \notin A_0$, that is, x is limit point of A_0 .

$\forall \delta > 0$, then $(B_\delta(x) \setminus \{x\}) \cap A_0 \neq \emptyset$

$\Rightarrow \exists t \in (B_\delta(x) \setminus \{x\}) \cap A_0$

$\Rightarrow t \in B_\delta(x) \setminus \{x\}$ and $t \in A_0$.

$\Rightarrow 0 < |t-x| < \delta$ and $p(t) = (1-t)a + tb \in A$ and $p(x) = (1-x)a + xb \in B$.

$\therefore d(p(t), p(x)) = |p(t) - p(x)| = |x-t| |a-b| \leq |x-t| \cdot (|a| + |b|) < \delta(|a| + |b|)$

and δ is arbitrary small

$\therefore p(x)$ is a limit point of A and $p(x) \in B \Rightarrow \overline{A} \cap B \neq \emptyset$. (contradiction) So $\overline{A_0} \cap B_0 = \emptyset$.

Similarly, $A_0 \cap \overline{B_0} = \emptyset$.

③ Since $p(1) = b \in B \Rightarrow 1 \in p^{-1}(B)$ and $p(0) = a \in A \Rightarrow 0 \in p^{-1}(A)$,

then $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$.

By ①, ②, ③, then A_0 and B_0 are separated.

(b)

Prove that $\exists t_0 \in (0, 1)$ s.t. $p(t_0) \notin A \cup B$.

(pf) Suppose $p(t) \in A \cup B$ for any $t \in [0, 1]$.

$$\Rightarrow p(0) \in A \text{ or } p(1) \in B \text{ for any } t \in [0, 1]$$

$$\Rightarrow t \in p(A) = A_0 \text{ or } t \in p(B) = B_0 \text{ for any } t \in [0, 1]$$

$$\Rightarrow t \in A_0 \cup B_0 \text{ for any } t \in [0, 1]$$

$$\Rightarrow [0, 1] \subseteq A_0 \cup B_0.$$

$$\text{Let } G = [0, 1] \cap A_0 \text{ and } H = [0, 1] \cap B_0. \text{ Then } G \cup H = [0, 1] \cap (A_0 \cup B_0) = [0, 1].$$

Since $0 \in A_0 \cap [0, 1]$ and $1 \in B_0 \cap [0, 1]$, then $G \neq \emptyset, H \neq \emptyset$.

Since $\overline{G} = \overline{[0, 1] \cap A_0} \subseteq \overline{[0, 1]} \cap \overline{A_0} = [0, 1] \cap \overline{A_0} = \emptyset$, then $\overline{G} \cap H = [0, 1] \cap \overline{A_0} \cap B_0 = \emptyset \Rightarrow \overline{G} \cap H = \emptyset$.

Since $\overline{H} = \overline{[0, 1] \cap B_0} \subseteq \overline{[0, 1]} \cap \overline{B_0} = [0, 1] \cap \overline{B_0} = \emptyset$, then $\overline{H} \cap G = [0, 1] \cap \overline{B_0} \cap A_0 = \emptyset \Rightarrow \overline{H} \cap G = \emptyset$.

So G and H are separated, but $[0, 1]$ is connected. (contradiction).

Thus $p(t_0) \notin A \cup B$ for some $t_0 \in [0, 1]$.

$\because t_0 = 1 \Rightarrow p(1) = b \in B \subseteq A \cup B \text{ and } t_0 = 0 \Rightarrow p(0) = a \in A \subseteq A \cup B$

& $t_0 \neq 0$ and $t_0 \neq 1$.

So $p(t_0) \notin A \cup B$ for some $t_0 \in (0, 1)$. ■

(c) Prove that every convex subset of \mathbb{R}^k is connected.

(pf) Let C be a convex set of \mathbb{R}^k .

Suppose C is not connected, that is, $\exists A, B$ are separated subsets of C with $A \cap B = \emptyset$,

$$C = A \cup B, \quad \overline{A} \cap B = \emptyset, \quad A \cap \overline{B} = \emptyset.$$

Choose $x \in A$ and $y \in B$. To define $p(t) = (1-t)x + ty$ for all $t \in \mathbb{R}$.

Since $p(0) = x \in A \subseteq C$ and $p(1) = y \in B \subseteq C$, then $x, y \in C$.

By definition of convex set, then $p(t) \in C$ for all $t \in [0,1]$. \diamond

By (b), then $\exists t_0 \in (0,1)$ s.t. $p(t_0) \notin A \cup B = C$. (contradiction)



Thus, C is connected.

