Name and Student ID's: _____

Homework 5, Advanced Calculus 1

1. Rudin Chapter 2 Exercise 18.
2. Rudin Chapter 2 Exercise 19cd.
3. Rudin Chapter 2 Exercise 20.
4. Rudin Chapter 2 Exercise 27.
5. Rudin Chapter 2 Exercise 28.

Solution: Let E be a closed subset of a separable metric space X. Let P be the set of condensation points of E, which is perfect by Exercise 27. It suffices to show that $P \subset E$, since then $P = E \cup (P \cap E^c)$ and the second set in the union is at most countable by Exercise 27.

By definition, condensation points are clearly limit point (uncountable set is clearly infinite), and therefore $P \subset E'$. But since E is closed, we have $E = E'$ and the proof is done.

6. Rudin Chapter 2 Exercise 29.

Solution: Note the the collection of open intervals

$$\mathcal{I} := \{(q - r, q + r) \mid q, r \in \mathbb{Q}\}$$

is a countable base for $\mathbb{R}$. (why?) Every open subset is therefore a union of some collection of intervals $\{I_i'\} \subset \mathcal{I}$. Since $\mathcal{I}$ is countable, there are at most countably many I_i:

$$E = \bigcup_{i=1}^{\infty} I_i'.$$

Partition $\mathbb{N}$ into a disjoint union of at most countably many subsets $\mathbb{N} = \cup_\alpha N_\alpha$, where $i, j \in N_\alpha$ for some α if $I_i' \cap I_j' \neq \phi$. Define

$$I_\alpha := \bigcup_{i \in N_\alpha} I_i'.$$

We may check that

- Each $I_\alpha = (a_\alpha, b_\alpha)$, where a_α (resp. b_α) is the infimum (resp. supremum) of the left (right) endpoints of open intervals in $\{I_i' \mid i \in N_\alpha\}$.
- $I_\alpha \cap I_\beta = \phi$ if $\alpha \neq \beta$.

Clearly, $E = \cup_\alpha I_\alpha$ is the desired union.

7. Rudin Chapter 2 Exercise 30.

Solution: Note that we may assume that all F_n are nonempty, since empty set does not contribute to the union. Suppose that $F_n^o = \emptyset$ for all n. Take $x_1 \in F_1$ and some ball $B_{r_1}(x_1)$ centered at x_1. By our assumption x_1 is not interior point of F_1.

8. Rudin Chapter 2 Exercise 19ab.

9. Rudin Chapter 2 Exercise 21.