Name and Student ID's: ______________________________

Homework 6, Advanced Calculus 1

!!! Please note the new group and classroom assignment !!!

1. Rudin Chapter 3 Exercise 24ab.

Solution: (a) For every Cauchy sequences $\{p_n\}, \{q_n\}$, we readily verify

- $\lim_{n\to\infty} d(p_n, p_n) = \lim_{n\to\infty} 0 = 0 \Rightarrow \{p_n\} \sim \{p_n\}$.
- $\{p_n\} \sim \{q_n\} \Rightarrow 0 = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n) \Rightarrow \{q_n\}\ \{p_n\}$.
- $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\} \Rightarrow \lim_{n\to\infty} d(p_n, r_n) \leq \lim_{n\to\infty} (d(p_n, q_n) + d(q_n, r_n)) = 0+0 = 0 \Rightarrow \{p_n\} \sim \{r_n\}$.

(b) Suppose $\{p_n\} \sim \{p_n'\}$ and $\{q_n\} \sim \{q_n'\}$, we have $\lim_{n\to\infty} d(p_n, p_n') = \lim_{n\to\infty} d(q_n, q_n') = 0$. Therefore

$$\lim_{n\to\infty} d(p_n, q_n) \leq \lim_{n\to\infty} d(p_n, p_n') + \lim_{n\to\infty} d(p_n', q_n') + \lim_{n\to\infty} d(q_n', q_n')$$

and the first, third terms are both zero due to their equivalence. So the function Δ is independent of choice of representatives and therefore defines a function on X^*. Since d is a metric, whose reflexivity and triangle inequality are preserved under limit operation, we see clearly that Δ is a metric on X^*. For the remaining parts of this problem, we will write P for both a Cauchy sequence and the equivalence class it belongs.

2. Rudin Chapter 3 Exercise 24c.

Solution: Let $\{P^m\}_m \subset X^*$ be Cauchy (in Δ). For all $\epsilon > 0$, there are two $\epsilon - N$ arguments we will utilize:

(i) Since $\{P^m\}_m$ is $\Delta-$ Cauchy, there exists M_ϵ such that

$$\Delta(P^m, P^r) = \lim_{k\to\infty} d(p_k^m, p_k^r) < \frac{\epsilon}{3}.$$

for all $m, r > M_\epsilon$. The statement can be rephrased as

$$\textit{There exists } M_\epsilon, K_{\epsilon,m,n} \in \mathbb{N} \textit{ so that } d(p_k^m, p_k^r) < \frac{\epsilon}{3} \ \ \forall m, r > M_\epsilon, k > K_{\epsilon,m,n}.$$

(ii) For all $m \in \mathbb{N}$, since $P^m = \{p_n^m\}$ is a Cauchy sequence in X, there exist N_m so that $d(p_n^m, p_l^m) < \frac{\epsilon}{3}$ for all $n, l > N_m$.

From the sequence of sequences above, take $P = \{p_{N_k}^k\}_k$, where N_k is chosen according to (ii) above. We must show that P is Cauchy in d and $\Delta(P^m, P) \to 0$ as $m \to \infty$. Again starting with an arbitrary $\epsilon > 0$, for $k > l > M_\epsilon$ in (i), we have

$$d(p_{N_k}^k, p_{N_l}^l) \leq d(p_{N_k}^k, p_v^k) + d(p_v^k, p_v^l) + d(p_v^l, p_{N_l}^l)$$

for any v. Take $v > max(N_k, N_l, K_{k,l}$, then according to the descriptions of (i) and (ii), the sum above is less than ϵ. Therefore, P is Cauchy in d.

To show that $P^m \to P$, we estimate

$$\Delta(P^m, P) = \lim_{k\to\infty} d(p_k^m, p_{N_k}^k).$$

Follow similar logics, we have

$$d(p_k^m, p_{N_k}^k) \le d(p_k^m, p_v^m) + d(p_v^m, p_v^k) + d(p_v^k, p_{N_k}^k).$$

For $m > M_\epsilon$, $k > max(M_\epsilon, N_m)$, and $v > max(M_\epsilon, N_m, N_k, K_{\epsilon,m,k})$, the above sum is then less than ϵ and result follows.

3. Rudin Chapter 3 Exercise 24d.

Solution: With the given constant (and therefore Cauchy) sequences P_p and P_q, it is clear from definition that $\Delta(P_p, P_q) = d(p, q)$. The surjective mapping $\varphi : X \to X^*$ given by $\varphi(p) = P_p$ is clearly injective. Since for all $p \neq q$ in X, we have $\Delta(P_p.P_q) = d(p, q) > 0$ and therefore $P_p \neq P_q$.

4. Rudin Chapter 3 Exercise 24e.

Solution: Take any $P \in X^*$ and $\epsilon > 0$, let the Cauchy sequence $\{p_k\}$ be a representative of P. Since it is Cauchy, there is N_ϵ so that $d(p_n, p_m) < \epsilon$ for all $n, m > N_\epsilon$. We then have

$$\Delta(P_{p_{N_\epsilon}}, P) = \lim_{k\to\infty} d(p_k, p_{N_\epsilon}) \le \epsilon$$

and therefore $\varphi(X)$ is dense in X^*. If X is complete, $\{p_k\}$ is convergent with limit p. Therefore

$$\Delta(P_p, P) = \lim_{k\to\infty} d(p, p_k) = 0$$

or $P_p = P$. Therefore $\varphi(X) = X^*$.

5. Rudin Chapter 3 Exercise 21.
6. Rudin Chapter 3 Exercise 22.
7. Rudin Chapter 3 Exercise 23.
8. Given $f : A \to B$ and $\{E_\alpha\}$ a collection of subsets of B, prove that
   (a) $f^{-1}(\cup_\alpha E_\alpha) = \cup_\alpha f^{-1}(E_\alpha)$.
   (b) $f^{-1}(\cap_\alpha E_\alpha) = \cap_\alpha f^{-1}(E_\alpha)$.
   (c) $f^{-1}(E_\alpha^c) = f^{-1}(E_\alpha)^c$.
9. Prove that 8a is still true with f^{-1} replaced by f, but 8b and 8c no longer hold.