

Hw6:

5. Rudin, ch3, #21.

If $\{E_n\}$ is a sequence of closed and bounded sets in a complete metric space X , if $E_n \supset E_{n+1}$, and if $\lim_{n \rightarrow \infty} \text{diam } E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.

(pf)

① Suppose $E_m = \emptyset$ for some m . Since $E_n \supset E_{n+1} \forall n \in \mathbb{N}$, then $E_n = \emptyset \forall n \geq m$.

Clearly, E_n is closed and bounded for $n \geq m$, $\lim_{n \rightarrow \infty} \text{diam } E_n = 0$, but $\bigcap_{n=1}^{\infty} E_n = \emptyset$.

Thus, this proposition false. So $E_n \neq \emptyset \forall n \in \mathbb{N}$.

② Choose $x_n \in E_n \forall n \in \mathbb{N}$. Claim: $\{x_n\}$ is a Cauchy sequence in X .

Given $\epsilon > 0$, since $\lim_{n \rightarrow \infty} \text{diam } E_n = 0$, then $\exists N \in \mathbb{N}$ s.t. $\text{diam } E_n < \epsilon \forall n \geq N$.

For $n, m \geq N$, $x_n \in E_n$, $x_m \in E_m$, then $x_n, x_m \in E_N$.

$\Rightarrow d(x_n, x_m) \leq \text{diam } E_N < \epsilon \quad \forall n, m \geq N$. So $\{x_n\}$ is Cauchy sequence.

③ Since X is complete, then $\lim_{n \rightarrow \infty} x_n = x$, for some $x \in X$.

Claim: $x \in E_n \forall n \in \mathbb{N}$.

Given $m \in \mathbb{N}$, since $\lim_{n \rightarrow \infty} x_n = x$, then this sequence $x_m, x_{m+1}, x_{m+2}, \dots$ converges to x .

Since, $\{x_m, x_{m+1}, x_{m+2}, \dots\} \subseteq E_m$ and E_m is closed, then $x \in E_m$.

Thus, $x \in E_n \forall n \in \mathbb{N}$. So $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$.

④ Suppose $x, y \in \bigcap_{n=1}^{\infty} E_n$ where $x \neq y$. Then $d(x, y) > 0$.

Choose $\varepsilon = \frac{1}{2} d(x, y) > 0$, then $\exists N_1 \in \mathbb{N}$ s.t. $\text{diam } E_n < \frac{1}{2} d(x, y)$ $\forall n \geq N_1$.

Since $x, y \in E_n \ \forall n \in \mathbb{N}$, then $x, y \in E_{N_1}$ and $d(x, y) \leq \text{diam } E_{N_1}$.

$\Rightarrow d(x, y) < \frac{1}{2} d(x, y)$ (contradiction).

Thus, $x = y$.

Therefore, $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point. ■

6. Rudin, ch3, ex2.

X is a complete metric space, $\{G_n\}$ is a sequence of dense open subsets of X .

Prove "Barres' theorem", that is, $\bigcap_{n=1}^{\infty} G_n$ is not empty. (In fact, it is dense in X .)

(pf)

Claim: $\bigcap_{n=1}^{\infty} G_n$ is dense in X .

Given $p \in X$ and $\varepsilon > 0$. Then $B_\varepsilon(p)$ is an open ball in X .

' G_1 is dense in X , then $B_\varepsilon(p) \cap G_1 \neq \emptyset$.

Let $x_1 \in B_\varepsilon(p) \cap G_1$. ' $B_\varepsilon(p) \cap G_1$ is open set, then $\exists r_1 > 0$ with $r_1 < 1$ such that

$$\overline{B_{r_1}(x_1)} \subseteq B_\varepsilon(p) \cap G_1. \quad \text{--- } \textcircled{1}$$

' G_2 is dense in X , then $B_{r_1}(x_1) \cap G_2 \neq \emptyset$.

Let $x_2 \in B_{r_1}(x_1) \cap G_2$. ' $B_{r_1}(x_1) \cap G_2$ is open set, then $\exists r_2 > 0$ with $r_2 < \frac{1}{2}$ such that

$$\overline{B_{r_2}(x_2)} \subseteq B_{r_1}(x_1) \cap G_2. \Rightarrow \overline{B_{r_2}(x_2)} \subseteq \overline{B_{r_1}(x_1)} \cap G_2 \subseteq B_\varepsilon(p) \cap G_1 \cap G_2. \quad \text{--- } \textcircled{2}$$

Step by step, then we have a sequence $\{x_n\}$ of X such that

$$\overline{B_{r_{n+1}}(x_{n+1})} \subseteq B_{r_n}(x_n) \cap G_{n+1} \text{ with } r_{n+1} < \frac{1}{n+1} \text{ for all } n \in \mathbb{N}.$$

$$\Rightarrow \overline{B_{r_{n+1}}(x_{n+1})} \subseteq \overline{B_{r_n}(x_n)} \cap G_{n+1} \subseteq \dots \subseteq B_\varepsilon(p) \cap G_1 \cap G_2 \cap \dots \cap G_{n+1}. \quad \text{--- } \textcircled{3}$$

$$\text{Now, } \lim_{n \rightarrow \infty} \text{diam } \overline{B_{r_n}(x_n)} = \lim_{n \rightarrow \infty} 2r_n < \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 \Rightarrow \lim_{n \rightarrow \infty} \text{diam } \overline{B_{r_n}(x_n)} = 0.$$

Since $\overline{B_{r_n}(x_n)}$ is closed, bounded $\forall n \in \mathbb{N}$, $\overline{B_{r_n}(x_n)} \supseteq \overline{B_{r_{n+1}}(x_{n+1})}$, $\lim_{n \rightarrow \infty} \text{diam } \overline{B_{r_n}(x_n)} = 0$,
in a complete metric space X

by ch3, Ex21, then $\bigcap_{n=1}^{\infty} \overline{B_{r_n}(x_n)} \neq \emptyset$.

Then $\phi \notin \bigcap_{n=1}^{\infty} \overline{B_{r_n}(x_n)} \subseteq \overline{B_r(x_1)} \subseteq B_\varepsilon(p) \cap G_1$,

$\phi \notin \bigcap_{n=1}^{\infty} \overline{B_{r_n}(x_n)} \subseteq \overline{B_{r_2}(x_2)} \subseteq B_\varepsilon(p) \cap G_1 \cap G_2$

⋮

⋮

$\phi \notin \bigcap_{n=1}^{\infty} \overline{B_{r_n}(x_n)} \subseteq \overline{B_{r_n}(x_n)} \subseteq B_\varepsilon(p) \cap G_1 \cap G_2 \cap \dots \cap G_n$

⋮

⋮

$\Rightarrow \phi \notin \bigcap_{n=1}^{\infty} \overline{B_{r_n}(x_n)} \subseteq B_\varepsilon(p) \cap \left[\bigcap_{n=1}^{\infty} G_n \right]$

$\hookrightarrow \bigcap_{n=1}^{\infty} G_n$ is dense in X . □

Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X .

Show that the sequence $\{d(p_n, q_n)\}$ converges.

(pf) For any $n, m \in \mathbb{N}$, use triangular inequality, then

$$\begin{aligned} d(p_n, q_n) &\leq d(p_n, p_m) + d(p_m, q_n) \\ &\leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n). \end{aligned}$$

$$\Rightarrow |d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n). \quad (1)$$

$$\text{Similarly, we have } d(p_m, q_m) \leq d(p_m, p_n) + d(p_n, q_m)$$

$$\leq d(p_m, p_n) + d(p_n, q_n) + d(q_n, q_m)$$

$$\Rightarrow |d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, q_n) + d(p_m, q_m) \quad \forall n, m \in \mathbb{N}. \quad (2)$$

By (1), (2), then $|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) \quad \forall n, m \in \mathbb{N}$.

Given $\varepsilon_{>0}$, since $\{p_n\}$ is Cauchy, then $\exists N_1 \in \mathbb{N}$ s.t. $d(p_m, p_n) < \varepsilon/2 \quad \forall n, m \geq N_1$,

since $\{q_n\}$ is Cauchy, then $\exists N_2 \in \mathbb{N}$ s.t. $d(q_m, q_n) < \varepsilon/2 \quad \forall n, m \geq N_2$.

Choose $N = \max\{N_1, N_2\} \in \mathbb{N}$, if $n, m \geq N$, then

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So $\{d(p_n, q_n)\}$ is Cauchy sequence in \mathbb{R} .

Since \mathbb{R} is complete, then $\{d(p_n, q_n)\}$ converges. ■

P. Given $f: A \rightarrow B$ and $\{E_\alpha\}$ a collection of subsets of B , prove that

$$(a) f^{-1}(\bigcup_{\alpha} E_{\alpha}) = \bigcup_{\alpha} f^{-1}(E_{\alpha})$$

$$(b) f^{-1}(\bigcap_{\alpha} E_{\alpha}) = \bigcap_{\alpha} f^{-1}(E_{\alpha})$$

$$(c) f^{-1}(E_{\alpha}^c) = f^{-1}(E_{\alpha})^c$$

(pf)

$$\textcircled{1} \quad \text{① } \forall x \in f^{-1}(\bigcup_{\alpha} E_{\alpha}) \Rightarrow f(x) \in \bigcup_{\alpha} E_{\alpha} \Rightarrow f(x) \in E_{\alpha} \text{ for some } \alpha$$

$$\Rightarrow x \in f^{-1}(E_{\alpha}) \text{ for some } \alpha$$

$$\Rightarrow x \in \bigcup_{\alpha} f^{-1}(E_{\alpha}).$$

$$\text{So } f^{-1}(\bigcup_{\alpha} E_{\alpha}) \subseteq \bigcup_{\alpha} f^{-1}(E_{\alpha})$$

$$\textcircled{2} \quad \text{② } \forall x \in \bigcup_{\alpha} f^{-1}(E_{\alpha}) \Rightarrow x \in f^{-1}(E_{\alpha}) \text{ for some } \alpha \Rightarrow f(x) \in E_{\alpha} \text{ for some } \alpha$$

$$\Rightarrow f(x) \in \bigcup_{\alpha} E_{\alpha} \Rightarrow x \in f^{-1}(\bigcup_{\alpha} E_{\alpha}).$$

$$\text{So } \bigcup_{\alpha} f^{-1}(E_{\alpha}) \subseteq f^{-1}(\bigcup_{\alpha} E_{\alpha})$$

$$\text{By } \textcircled{1}, \textcircled{2}, \text{ then } f^{-1}(\bigcup_{\alpha} E_{\alpha}) = \bigcup_{\alpha} f^{-1}(E_{\alpha}).$$

$$\textcircled{3} \quad \text{① } \forall x \in f^{-1}(\bigcap_{\alpha} E_{\alpha}) \Rightarrow f(x) \in \bigcap_{\alpha} E_{\alpha} \Rightarrow f(x) \in E_{\alpha} \text{ for all } \alpha$$

$$\Rightarrow x \in f^{-1}(E_{\alpha}) \text{ for all } \alpha \Rightarrow x \in \bigcap_{\alpha} f^{-1}(E_{\alpha})$$

$$\text{So } f^{-1}(\bigcap_{\alpha} E_{\alpha}) \subseteq \bigcap_{\alpha} f^{-1}(E_{\alpha})$$

$$\textcircled{4} \quad \text{② } \forall x \in \bigcap_{\alpha} f^{-1}(E_{\alpha}) \Rightarrow x \in f^{-1}(E_{\alpha}) \text{ for all } \alpha \Rightarrow f(x) \in E_{\alpha} \text{ for all } \alpha \Rightarrow f(x) \in \bigcap_{\alpha} E_{\alpha}$$

$$\Rightarrow x \in f^{-1}(\bigcap_{\alpha} E_{\alpha})$$

$$\text{So } \bigcap_{\alpha} f^{-1}(E_{\alpha}) \subseteq f^{-1}(\bigcap_{\alpha} E_{\alpha}) \text{ By } \textcircled{3}, \textcircled{4}, \text{ then } f^{-1}(\bigcap_{\alpha} E_{\alpha}) = \bigcap_{\alpha} f^{-1}(E_{\alpha}).$$

(C)

$$\textcircled{1} \quad \forall x \in \tilde{f}(E_\alpha^c) \Rightarrow f(x) \in E_\alpha^c \Rightarrow f(x) \notin E_\alpha \Rightarrow x \notin \tilde{f}(E_\alpha) \Rightarrow x \in \tilde{f}(E_\alpha)^c$$

$$\textcircled{2} \quad \text{So } \tilde{f}(E_\alpha^c) \subseteq \tilde{f}(E_\alpha)^c$$

$$\textcircled{2} \quad \forall x \in \tilde{f}(E_\alpha)^c \Rightarrow x \notin \tilde{f}(E_\alpha) \Rightarrow f(x) \notin E_\alpha \Rightarrow f(x) \in E_\alpha^c \Rightarrow x \in \tilde{f}(E_\alpha^c)$$

$$\text{So } \tilde{f}(E_\alpha)^c \subseteq \tilde{f}(E_\alpha^c)$$

$$\text{By } \textcircled{1}, \textcircled{2}, \text{ then } \tilde{f}(E_\alpha^c) = \tilde{f}(E_\alpha)^c.$$



9. Prove that $\varphi(a)$ is still true with f' replaced by f , but $\varphi(b)$ and $\varphi(c)$ no longer hold.

$$\text{(pf)} \quad \text{Claim: } f(\bigcup_{\alpha} E_{\alpha}) = \bigcup_{\alpha} f(E_{\alpha}).$$

① $\forall y \in f(\bigcup_{\alpha} E_{\alpha})$, then $\exists x \in \bigcup_{\alpha} E_{\alpha}$ s.t. $y = f(x)$.

$\Rightarrow y = f(x)$ where $x \in E_{\alpha}$ for some α

$\Rightarrow y \in f(E_{\alpha})$ for some $\alpha \Rightarrow y \in \bigcup_{\alpha} f(E_{\alpha})$

$$\therefore f(\bigcup_{\alpha} E_{\alpha}) \subseteq \bigcup_{\alpha} f(E_{\alpha})$$

② $\forall y \in \bigcup_{\alpha} f(E_{\alpha}) \Rightarrow y \in f(E_{\alpha})$ for some $\alpha \Rightarrow y = f(x)$ for some $x \in E_{\alpha}$ for some α

$\Rightarrow y = f(x)$ where $x \in \bigcup_{\alpha} E_{\alpha}$

$\Rightarrow y \in f(\bigcup_{\alpha} E_{\alpha}) \quad \therefore \bigcup_{\alpha} f(E_{\alpha}) \subseteq f(\bigcup_{\alpha} E_{\alpha})$

By ①, ②, then $f(\bigcup_{\alpha} E_{\alpha}) = \bigcup_{\alpha} f(E_{\alpha})$. ■

(pf)

$$\text{Claim: } f(\bigcap_{\alpha} E_{\alpha}) \subseteq \bigcap_{\alpha} f(E_{\alpha})$$

$\because \bigcap_{\alpha} E_{\alpha} \subseteq E_{\alpha}$ for all $\alpha \Rightarrow f(\bigcap_{\alpha} E_{\alpha}) \subseteq f(E_{\alpha})$ for all $\alpha \Rightarrow f(\bigcap_{\alpha} E_{\alpha}) \subseteq \bigcap_{\alpha} f(E_{\alpha})$. ■

But $f(\bigcap_{\alpha} E_{\alpha}) \neq \bigcap_{\alpha} f(E_{\alpha})$

$$f(x) = \begin{cases} 1, & x \neq 0, \\ 2, & x = 0. \end{cases} \quad E_n = \left(\frac{1}{n}, \frac{1}{n} \right) \text{ for all } n \in \mathbb{N} \Rightarrow f(E_n) = \{1, 2\} \quad \text{Then } \bigcap_{n=1}^{\infty} f(E_n) = \{1, 2\}.$$

$$\bigcap_{n=1}^{\infty} E_n = \{0\} \Rightarrow f\left(\bigcap_{n=1}^{\infty} E_n\right) = \{2\}$$

But $f\left(\bigcap_{n=1}^{\infty} E_n\right) \neq \bigcap_{n=1}^{\infty} f(E_n)$. ※

$\left\langle \text{sol} \right\rangle$

$$f(E_\alpha^c) \neq f(E_\alpha)^c$$

$$Ex_2 \\ f(x) = \begin{cases} 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

$$E = [-1, 1] \Rightarrow f(E) = \{1, 2\} \Rightarrow f(E)^c = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$$

$$E^c = (-\infty, -1) \cup (1, \infty) \Rightarrow f(E^c) = \{1\}$$

$$\text{So } f(E)^c \neq f(E^c).$$

X