Name and Student ID's: ______________________________

Homework 7, Advanced Calculus 1

1. Rudin Chapter 4 Exercise 20a.
2. Rudin Chapter 4 Exercise 20b.
3. Rudin Chapter 4 Exercise 21.
4. Rudin Chapter 4 Exercise 22.

The *Cantor Function* is defined by $f : [0,1] \to [0,1]$ with the following rules. Recall that every $x \in [0,1]$ can be written in *tertiary* expression $x = \sum_j a_j 3^{-j}$, with $a_j = 0, 1, 2$. The expression is unique except that

$$\sum_{j=1}^{N-1} a_j 3^{-j} + a_N 3^{-N} + \sum_{j=N+1}^{\infty} 2 \cdot 3^{-j} = \sum_{j=1}^{N-1} a_j 3^{-j} + (a_N + 1) 3^{-N}.$$

We pick the first expression to ensure uniqueness. The Cantor set $C \subset [0,1]$ is defined by those real numbers with $a_j \neq 1\ \forall j$. We define f separately on C and C^c. For $x = \sum_j a_j 3^{-j} \in C$, we define

$$f(x) = \sum_j \frac{a_j}{2} 2^{-j}.$$

For $x = \sum_j a_j 3^{-j} \in C^c$, we define

$$f(x) = \sum_{j=1}^{J_x - 1} \frac{a_j}{2} 2^{-j} + 2^{-J_x},$$

where J_x is the first digit of x with $a_j = 1$.

Problems 5,6 concern the Cantor function and related topics.

5. Prove that the Cantor function f is uniformly continuous on $[0,1]$ and differentiable on C^c.
6. Prove that there exist constants $C, \alpha > 0$ so that

$$|f(x) - f(y)| \leq C|x - y|^{\alpha} \ \ \forall x, y \in [0,1].$$

7. Functions satisfying the condition in Problem 6 on its domain is said to be *Hölder continuous* with exponent α. Prove that

$$\{\textit{Hölder continuous function}\} \subsetneq \{\textit{Uniformly Continuous Functions}\}.$$

8. Rudin Chapter 4 Exercise 25.

9. Rudin Chapter 4 Exercise 26.