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Homework 7, Advanced Calculus 1

1. Rudin Chapter 4 Exercise 20a.

Solution:

Suppose that $\rho_E(x) = 0 = \inf_{z\in E} d(x,z)$. For all $\epsilon > 0$, by definition of infimum, there is $z_\epsilon \in E$ so that $0 \le d(x, z_\epsilon) < \epsilon$ and therefore $x \in \bar{E}$. Conversely, $x \in \bar{E}$ implies that the set of $\{d(x,z)\}_{z\in E}$ consist of nonnegative numbers that are arbitrarily small, and therefore the infimum must be 0.

2. Rudin Chapter 4 Exercise 20b.

Solution:

Given $x, y \in X$, for every $z \in E$, we have $\rho_E(x) \le d(x,z) \le d(x,y) + d(y,z)$. That is $\rho_E(x) - d(x,y) \le d(y,z)$ for all $z \in E$, or $\rho_E(x) - d(x,y)$ is a lower bound for $\{d(y,z)\}_{z\in E}$. Therefore, $\rho_E(x) - d(x,y) \le \inf\{d(y,z)\}_{z\in E} = \rho_E(y)$, or $\rho_E(x) - \rho_E(y) \le d(x,y)$. Exchanging x and y, we get $\rho_E(y) - \rho_E(x) \le d(y,x) = d(x,y)$. Therefore, $|\rho_E(x) - \rho_E(y)| \le d(x,y)$ and result follows.

3. Rudin Chapter 4 Exercise 21.

Solution: By Exercise 20, $\rho_K : K \to F$ is a continuous function on X and therefore on the compact set K. Therefore ρ_F attains a minimum on K:

$$\rho_F(x_0) = \inf_{x\in K} \rho_F(x).$$

It then suffices to show that $\rho_F(x) > 0 \ \forall x \in K$, which follows easily from Exercise 20. Indeed, if $\rho_F(x) = 0$ for some $x \in K$, then $x \in \bar{F}$. But since F is closed, we have $\bar{F} = F \Rightarrow x \in K \cap F$, which is a contradiction.

4. Rudin Chapter 4 Exercise 22.

Solution:

Since ρ_A and ρ_B are both continuous function, f is continuous except at point p where $\rho_A(p) + \rho_B(p) = 0$. But since both functions are nonnegative, it only happens when $\rho_A(p) = \rho_B(p) = 0$, or $p \in \bar{A} \cap \bar{B}$. However, since both sets are closed, we have $x \in A \cap B$, contradicting the fact $A \cap B = \emptyset$.

$f(p) = 0$ precisely when $\rho_A(p) = 0$ and $\rho_A(p) + \rho_B(p) > 0$, which are true iff $p \in \bar{A} = A$. $f(p) = 1 \Leftrightarrow \rho_A(p) = \rho_A(p) + \rho_B(p) \Leftrightarrow \rho_B(p) = 0 \Leftrightarrow p \in \bar{B} = B$.

It is clear that $A = f^{-1}(0) \subset V = f^{-1}([0, \frac{1}{2}))$ and similarly $B \subset W$. The openness of V and W follow from the fact that $0 \le f(X) \le 1$ and therefore $V = f^{-1}([0, \frac{1}{2})) = f^{-1}((-\frac{1}{2}, \frac{1}{2}))$ and $W = f^{-1}((-\frac{1}{2}, 1]) = f^{-1}((\frac{1}{2}, 2))$ plus the continuity of f.

The *Cantor Function* is defined by $f : [0,1] \to [0,1]$ with the following rules. Recall that every $x \in [0,1]$ can be written in *tertiary* expression $x = \sum_j a_j 3^{-j}$, with $a_j = 0, 1, 2$. The expression is unique except that

$$\sum_{j=1}^{N-1} a_j 3^{-j} + a_N 3^{-N} + \sum_{j=N+1}^{\infty} 2\cdot 3^{-j} = \sum_{j=1}^{N-1} a_j 3^{-j} + (a_N+1)3^{-N}.$$

We pick the first expression to ensure uniqueness. The Cantor set $C \subset [0,1]$ is defined by those real numbers with $a_j \neq 1\ \forall j$. We define f separately on C and C^c. For $x = \sum_j a_j 3^{-j} \in C$, we define

$$f(x) = \sum_j \frac{a_j}{2} 2^{-j}.$$

For $x = \sum_j a_j 3^{-j} \in C^c$, we define

$$f(x) = \sum_{j=1}^{J_x-1} \frac{a_j}{2} 2^{-j} + 2^{-J_x},$$

where J_x is the first digit of x with $a_j = 1$.

Problems 5,6 concern the Cantor function and related topics.

5. Prove that the Cantor function f is uniformly continuous on $[0,1]$ and differentiable on C^c.

Solution: Continuity follows clearly from Problem 6 and 7 below. It is differentiable on C^c since every $x \in C^c$ is contained in an open interval on which f is constant. It is therefore differentiable at x with $f'(x) = 0$.

6. Prove that there exist constants $K, \alpha > 0$ so that

$$|f(x) - f(y)| \leq K|x-y|^\alpha \quad \forall x, y \in [0,1]. \tag{1}$$

Solution: First we check that f is monotonic. Given $y = \sum_j a_j 3^{-j} < x = \sum_j b_j 3^{-j}$, there is $N \in \mathbb{N}$ so that $a_N < b_N$ (ie. $(0,1),(1,2),(0,2)$) and $a_j = b_j\ \forall j < N$. By definition of f, the first $N-1$ digits of $f(x)$ and $f(y)$ are still the same and the N^{th} digit of $f(y)$ is less or equal to that of $f(x)$. The digits after the N^{th} are of course irrelevant to the order. Therefore $f(y) \leq f(x)$ and the function is monotonically increasing.

The inequality above is certainly true if $x = y$. Take $y < x$, with the expression in the first paragraph. There are four possibilities:

- $x, y \in C$.
- $x \in C, y \in C^c$.
- $x \in C^c, y \in C$.
- $x, y \in C^c$.

We first obverse that it suffices to prove the first case. Indeed, for every $t = \sum_j a_j 3^{-j} \in C^c$, let a_N be the first digit with $a_N = 1$. Then there are $t_1 < t < t_2$ with $t_1, t_2 \in C$ and $f(t_1) = f(t) = f(t_2)$. t_1 and t_2 are the two endpoints of the open interval t belongs to. Explicitly, $t_1 = \sum_{j=1}^{N-1} a_j 3^{-j} + \sum_{N+1}^{\infty} 2\cdot 3^{-j}$ and $t_2 = \sum_{j=1}^{N-1} a_j 3^{-j} + 2\cdot 3^{-N}$. With these observations, for every $y < x$ in $[0,1]$, there exist then $y' \le x'$, both in C, so that $f(x) = f(x'), f(y) = f(y')$, and $|x' - y'| \le |x - y|$. Therefore, if

$$|f(x') - f(y')| \le C|x' - y'|^\alpha,$$

we certainly have

$$|f(x) - f(y)| \le C|x - y|^\alpha.$$

We now prove the first case. For $y = \sum_j a_j 3^{-j} < x = \sum_j b_j 3^{-j}$, since $x, y \in C$, we $a_N = 0 < b_N = 2$. Then,

$$\begin{aligned} |f(x) - f(y)| &= 2^{-N} + \sum_{j=N+1}^{\infty} \frac{b_j - a_j}{2} 2^{-j} \\ &\le 2^{-N} + \sum_{j=N+1}^{\infty} 2^{-j} \\ &= 2^{-N+1}. \end{aligned}$$

On the other hand

$$|x - y| = 2 \cdot 3^{-N} + \sum_{j=N+1}^{\infty} (b_j - a_j) 3^{-j} \ge 2 \cdot 3^{-N} - \sum_{j=N+1}^{\infty} 2 \cdot 3^{-j} = 3^{-N}.$$

Therefore, take $\alpha = \log_3 2$, we have $|x - y|^\alpha \ge 2^{-N}$ and

$$\frac{|f(x) - f(y)|}{|x - y|^\alpha} \le 2,$$

which is the desired inequality.

7. Functions satisfying the condition in Problem 6 on its domain is said to be *Hölder continuous* with exponent α. Prove that

$$\{\textit{Hölder continuous function}\} \subsetneq \{\textit{Uniformly Continuous Functions}\}.$$

Solution: The inclusion is easy. Let f satisfies (1) in Problem 6. For all ϵ, take $\delta = \frac{\epsilon^\alpha}{K}$ then uniform continuity follows.

For the properness of this inclusion, consider

$$f(x) = \begin{cases} \frac{1}{\log x} \; ; \; x \in (0, \frac{1}{2}] \\ 0 \; ; \; x = 0. \end{cases} \tag{2}$$

The function is continuous on $[0, \frac{1}{2}]$, and therefore uniformly continuous since $[0, \frac{1}{2}]$ is compact. However, for any $\alpha > 0$, by L'Hospital's rule, we have

$$\lim_{x \to 0} \frac{\frac{1}{\log x}}{x^\alpha} = \infty$$

and therefore (1) is impossible with $y = 0$.

8. Rudin Chapter 4 Exercise 25.

Solution:

(a) Note that the distance we use here is the Euclidean distance $d(a, b) = \|a - b\|$.

Following the hint, we show that $(K+C)^c$ is open. Let $z \in (K+C)^c$ and consider $F = z - C := \{z - c \mid c \in C\}$. It is clear that $F \cap K = \emptyset$. Since if $x \in F \cap K, x = z - y$ for some $y \in C$, then $z = y + x \in C + K$, contradicting $z \in (K + C)^c$. It is also clear that F is closed, since if $x \in F', z - x \in C' \subset C$ as C is closed. Therefore $x \in F$.

We now have a closed set F and a compact set K disjoint from each other. By Exercise 21, there is $\delta > 0$ so that $d(p, q) > \delta$ for all $p \in K, q \in F$. We finally show that

$$B_\delta(z) \cap (K + C) = \emptyset$$

and the openness of $(K + C)^c$, or the closedness of $K + C$ follows. To the contrary, suppose that we have $a + b \in B_\delta(z) \cap (K + C)$, where $a \in K, b \in C$. Since $a + b \in B_\delta(z)$, we have

$$\|a + b - z\| = d(a + b, z) < \delta.$$

On the other hand, since $b \in C$, we have $z - b \in F$. With $a \in K$, we have

$$\|a + b - z\| = \|a - (z - b)\| = d(a, z - b) > \delta.$$

The two inequalities contradict each other and we have $B_\delta(z) \cap (K + C) = \emptyset$.

(b) $C_1 = \mathbb{Z}$ and $C_2 = \{m\alpha \mid m \in \mathbb{Z}\}$ for some $\alpha \notin \mathbb{Q}$. Both sets are closed since both of them have no limit point. We show that $C = C_1 + C_2$ is not closed. The result follows if we show that C is dense in $\mathbb{R}$. This is true because C is clearly countable. If it is dense, then it can not be closed. Otherwise, we have

$$\mathbb{R} = \overline{C} = C$$

but the right hand side is countable while the left hand side is not.

We now show that C is dense. For any $x \in \mathbb{R}$, define $(x) = x - [x]$ to be the *fractional part* of x. Here, $[x]$ is the largest integer $\leq x$. Since $\alpha \notin \mathbb{Q}$, we have $(m\alpha) \neq (m'\alpha)$ if m, m' are two distinct integers. Indeed, if $(m\alpha) = (m'\alpha)$ while $m \neq m'$, we have $m\alpha - [m\alpha] = m'\alpha - [m'\alpha]$. But then $\alpha = \frac{[m\alpha] - [m'\alpha]}{m - m'} \in \mathbb{Q}$, a contradiction.

For each n, divide $[0, 1]$ into n subintervals of length $\frac{1}{n}$. By the discussion above, $\{(j\alpha)\}_{j=1}^n$ is a set of n distinct values in $[0, 1]$. By pigeonhole principle, two of these values, say $(m\alpha) < (m'\alpha)$ belong to one subinterval. Their distance is then less than $\frac{1}{n}$, and we have

$$0 < (m'\alpha) - (m\alpha) = m'\alpha - [m'\alpha] - m\alpha + [m\alpha] = (m' - m)\alpha + ([m'\alpha] - [m\alpha]) < \frac{1}{n}.$$

Let $\alpha_n = (m' - m)\alpha + ([m'\alpha] - [m\alpha]) \in (0, \frac{1}{n})$. For any $x \in [0, 1]$, it is now clear that $x \in (r\alpha_n, (r+1)\alpha_n)$ for some integer r, and the length of the interval is less than $\frac{1}{n}$. Finally, for $x \in \mathbb{R}$, we have $x = [x] + (x)$. $[x] \in \mathbb{Z}$ and $(x) \in [0, 1]$. For all $n \in \mathbb{N}$, there is $\alpha_n \in C$ so that $|(x) - \alpha_n| < \frac{1}{n}$. Then $[x] + \alpha_n \in C$ and $|[x] + \alpha_n - x| < \frac{1}{n}$.

9. Rudin Chapter 4 Exercise 26.

Solution: Since $g : Y \to Z$ is one-to-one, $g : Y \to g(Y)$ is surjective and we have $g^{-1} : g(Y) \to Y$. By Theorem 4.17, g^{-1} is continuous. The continuity of f follows easily by the observation that $f = g^{-1} \circ h$ and both functions on the right are continuous.

In addition, since g is continuous and Y is compact, $g(Y)$ is compact and therefore $g^{-1} : g(Y) \to Y$ is uniformly continuous. Therefore, if h is uniformly continuous, so is f.

If Y is no longer compact but X and Z are, the statement fails by considering the following counterexample. $X = [0, 1], Y = [0, \frac{\pi}{2})$, and Z be the unit circle in $\mathbb{R}^2$. Let $f : X \to Y$ be defined by

$$f(x) = \begin{cases} 0 \; ; \; x = 0 \\ \tan^{-1}(-\log x) \; ; \; x \in (0, 1). \end{cases} \tag{3}$$

$g(t) = (\cos(4t), \sin(4t))$. Then $h = f \circ g : [0, 1] \to S^1$ is continuous, and therefore uniformly continuous. g is continuous and one-to-one, however, f is not even continuous.