

Fudh, ch7, *1

1. prove that every uniformly convergent sequence of bounded function is uniformly bounded.

(pf)

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a uniformly convergent sequence of bounded functions.

Let $|f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$, for any x .

Given $\epsilon = 1 > 0$, since $\{f_n\}$ is uniformly convergent sequence, then $\exists N \in \mathbb{N}$ s.t.

if $m, n \geq N$, we have $|f_n(x) - f_m(x)| < 1 \quad \forall n, m \geq N$, for any x .

As $m \geq N$, then $|f_m(x)| \leq |f_m(x) - f_N(x)| + |f_N(x)| < 1 + M_N$ for any x .

Let $M = 1 + \max\{M_1, M_2, \dots, M_N\}$.

Then $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$, for any x . (uniformly bounded).



2. If $\{f_n\}$ and $\{g_n\}$ converges uniformly on a set E , prove that $\{f_n + g_n\}$ converges uniformly on E . If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequence of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E .

(pf) ① Let $f_n \xrightarrow{\text{uniformly}} f$ on E and $g_n \xrightarrow{\text{uniformly}} g$ on E .

Given $\varepsilon > 0$, since $f_n \xrightarrow{\text{uniformly}} f$ on E , then $\exists N_1 \in \mathbb{N}$ s.t.

$|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for $n \geq N_1$ for any $x \in E$.

Since $g_n \xrightarrow{\text{uniformly}} g$ on E , then $\exists N_2 \in \mathbb{N}$ s.t.

$|g_n(x) - g(x)| < \frac{\varepsilon}{2}$ for $n \geq N_2$ for any $x \in E$.

Choose $N_3 = \max\{N_1, N_2\} \in \mathbb{N}$, if $n \geq N_3$, then we have

$$|(f_n(x) + g_n(x)) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon, \text{ for all } x \in E.$$

So $\{f_n + g_n\} \xrightarrow{\text{uniformly}} f + g$ on E .

② Since f_n and g_n are bounded for all $n \in \mathbb{N}$, then $\exists M > 0 \in \mathbb{R}$ s.t. $|f_n| \leq M, |g_n| \leq M$ on E for all $n \in \mathbb{N}$.

Given $\varepsilon > 0$, since $f_n \xrightarrow{\text{uniformly}} f$ on E , then $\exists N_3 \in \mathbb{N}$ s.t.

$|f_n(x) - f(x)| < \frac{\varepsilon}{2M}$ for $n \geq N_3$, for any $x \in E$.

Since $g_n \xrightarrow{\text{uniformly}} g$ on E , then $\exists N_4 \in \mathbb{N}$ s.t.

$|g_n(x) - g(x)| < \frac{\varepsilon}{2M}$ for $n \geq N_4$, for any $x \in E$.

Choose $N_4 = \max\{N_3, N_4\} \in \mathbb{N}$.

$$\left| f_n(x) \cdot g_n(x) - f(x) \cdot g(x) \right| \leq |f_n(x) - f(x)| \cdot |g_n(x)| + |f(x)| \cdot |g_n(x) - g(x)| \text{ for all } x \in E.$$

$|f_n(x)| \leq M$ for all $n \in \mathbb{N}$, for any $x \in E$, then $|f(x)| \leq M$ for any $x \in E$.

$$\begin{aligned} \left| f_n(x) \cdot g_n(x) - f(x) \cdot g(x) \right| &\leq |f_n(x) - f(x)| \cdot |g_n(x)| + |f(x)| \cdot |g_n(x) - g(x)| \\ &< \frac{\varepsilon}{2M} \cdot M + \frac{\varepsilon}{2M} \cdot M = \varepsilon, \text{ if } n \geq N_4, \text{ for all } x \in E. \end{aligned}$$

So $f_n \cdot g_n \xrightarrow{\text{uniformly}} f \cdot g$ on E . □

3. Consider sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E , but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

<Sol>

Let $f_n(x) = x$ for any $x \in \mathbb{R}$ for all $n \in \mathbb{N}$. (unbounded for all $n \in \mathbb{N}$)

Let $g_n(x) = \frac{1}{n}$ for any $x \in \mathbb{R}$ for all $n \in \mathbb{N}$. (bounded for all $n \in \mathbb{N}$)

Clearly, $f_n(x) \xrightarrow{\text{uniformly}} f(x) = x \quad \forall x \in \mathbb{R}$ and $g_n(x) \xrightarrow{\text{uniformly}} g(x) = 0 \quad \forall x \in \mathbb{R}$.

Now, $f_n(x) g_n(x) = \frac{x}{n}$ for any $x \in \mathbb{R}$ for all $n \in \mathbb{N}$.

$$\lim_{n \rightarrow \infty} f_n(x) g_n(x) = \lim_{n \rightarrow \infty} \frac{x}{n} = 0 \quad \text{for any } x \in \mathbb{R}.$$

Let $h(x) = 0$ for any $x \in \mathbb{R}$. Then $\lim_{n \rightarrow \infty} f_n(x) g_n(x) = h(x)$ for any $x \in \mathbb{R}$.

For each $n \in \mathbb{N}$, choose $x = n \in \mathbb{R}$, then $f_n(x) g_n(x) = 1$ for each $n \in \mathbb{N}$.

So $\{f_n g_n\}$ does not converge uniformly on \mathbb{R} . ◻

$$4. f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous whenever the series converges? Is f bounded?

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$$\text{① } x=0, f(0) = \sum_{n=1}^{\infty} \frac{1}{1} = +\infty. (\Rightarrow f \text{ is not bounded.})$$

$x = \frac{1}{n^2}$, $f\left(\frac{1}{n^2}\right)$ is undefined for all $n \in \mathbb{N}$, then $f\left(\frac{1}{n^2}\right)$ is undefined for all $n \in \mathbb{N}$.

For any $x \in \mathbb{R} \setminus \{0, \frac{1}{n^2} | n \in \mathbb{N}\}$, then $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ is defined.

$$\text{② } \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \leq \sum_{n=1}^{\infty} \frac{1}{n^2x} = \frac{1}{x} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Choose $\delta > 0$.

Case 1: Let $x \in [\delta, \infty) \setminus \{\frac{1}{n^2} | n \in \mathbb{N}\}$.

$\because \sum_{n=1}^{\infty} \left| \frac{1}{1+n^2x} \right| \leq \sum_{n=1}^{\infty} \frac{1}{\delta n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{\delta n^2} = \frac{1}{\delta} \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converge, by thm 7.10,}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges uniformly on $[\delta, \infty)$.

Case 2: Let $x \in (-\infty, -\delta] \setminus \{\frac{1}{n^2} | n \in \mathbb{N}\}$. Then $x < -\delta < -\frac{1}{n_0^2}$ for some $n_0 \in \mathbb{N}$.

$$\left| \sum_{n=n_0}^{\infty} \frac{1}{1+n^2x} \right| = \sum_{n=n_0}^{\infty} \frac{1}{n^2} \left| \frac{1}{x + \frac{1}{n^2}} \right| \\ = \sum_{n=n_0}^{\infty} \frac{1}{n^2} \cdot \frac{1}{-\delta - \frac{1}{n^2}} \leq \sum_{n=n_0}^{\infty} \frac{1}{n^2} \cdot \frac{1}{-\delta - \frac{1}{n_0^2}} \leq \sum_{n=n_0}^{\infty} \frac{1}{\delta n^2} \text{ converges, by thm 7.10,}$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges uniformly on $(-\infty, -\delta]$.

(3)

For $x \in (-\delta, \delta) \setminus \{0, \frac{1}{n^2} \mid n \in \mathbb{N}\}$.

Suppose $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges uniformly for any $x \in (-\delta, \delta) \setminus \{0, \frac{1}{n^2} \mid n \in \mathbb{N}\}$.

case 1: $x \in (0, \delta)$. Since $\frac{1}{1+n^2x}$ is bounded on $(0, \delta)$, by ch7, then $f(x)$ is bounded.

But $f(\frac{1}{m^2}) = \sum_{n=1}^{\infty} \frac{1}{1+\frac{n^2}{m^2}} \geq \sum_{n=1}^m \frac{1}{1+\frac{n^2}{m^2}} \geq \frac{m}{2}$ for which $\frac{1}{m^2} < \delta$, $m \in \mathbb{N}$.

$\Rightarrow f$ is not bounded on $(0, \delta)$. (contradiction) So f_n does not uniformly converge on $(0, \delta)$.

case 2: $x \in (-\delta, 0) \setminus \{\frac{1}{n^2} \mid n \in \mathbb{N}\}$. ($\because \lim_{x \rightarrow 0^-} f(x) = +\infty$, $f(0)$ is not bounded above.)

Let $S_N(x) = \sum_{n=1}^N \frac{1}{1+n^2x}$ for any $x \in (-\delta, 0)$ for each $N \in \mathbb{N}$.

Since $|S_N(x) - S_{N+1}(x)| = \left| \frac{1}{1+N^2x} \right|$ and $x = \frac{-1}{2N_0^2} \in (-\delta, 0)$ for some $N_0 \in \mathbb{N}$,
 $(\Rightarrow \frac{1}{2N^2} > \frac{1}{2N_0^2} \text{ for all } N \geq N_0)$

then $|S_N(\frac{-1}{2N^2}) - S_{N+1}(\frac{-1}{2N^2})| = 2$ for all $N \geq N_0$. (contradiction)

\Rightarrow So f_n does not uniformly converge on $(-\delta, 0)$.

By case 1, case 2, then f_n does not converge uniformly on $(-\delta, \delta) \setminus \{0, \frac{1}{n^2} \mid n \in \mathbb{N}\}$

(4)

Since $\frac{1}{1+n^2x}$ is continuous on $\mathbb{R} \setminus \{\frac{1}{n^2} \mid n \in \mathbb{N}\}$, f_n converges uniformly on $(-\infty, \delta] \cup [\delta, \infty)$,

then $f(x)$ is continuous on $(\mathbb{R} \setminus \{\frac{1}{n^2} \mid n \in \mathbb{N}\}) \cap [(-\infty, -\delta] \cup [\delta, \infty)]$, for any $\delta > 0$.

So $f(x)$ is continuous on $\mathbb{R} \setminus (\{\frac{1}{n^2} \mid n \in \mathbb{N}\} \cup \{0\})$.

Hw 8.

Rudin, ch 7, #5.

$$f_n(x) = \begin{cases} 0 & , x < \frac{1}{n+1} \\ \sin \frac{\pi x}{2} & , \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & , x > \frac{1}{n} \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x does not imply uniform convergence.

(f) ① Claim: $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in \mathbb{R}$.

Case 1: if $x \leq 0$ or $x \geq 1$.

Given $x \leq 0$. For each $n \in \mathbb{N}$, then $x \leq 0 < \frac{1}{n+1} \Rightarrow f_n(x) = 0 \quad \forall n \in \mathbb{N} \Rightarrow \lim_{n \rightarrow \infty} f_n(x) = 0$.

Given $x \geq 1$. For each $n \in \mathbb{N}, n \geq 2$, then $x \geq 1 > \frac{1}{n} \Rightarrow f_n(x) = 0 \quad \forall n \geq 2 \Rightarrow \lim_{n \rightarrow \infty} f_n(x) = 0$.

Case 2: if $0 < x < 1$.

Given $0 < x < 1$. Then $\frac{1}{n_0} \leq x$ for some $n_0 \in \mathbb{N} \Rightarrow \frac{1}{n} \leq \frac{1}{n_0} \leq x$ for all $n \geq n_0$

$\Rightarrow f_n(x) = 0$ for all $n \geq n_0$

$\Rightarrow \lim_{n \rightarrow \infty} f_n(x) = 0$.

Thus, $f_n(x) \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in \mathbb{R}$.

Define: $f(x) = 0$ for all $x \in \mathbb{R}$.

Then $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$ and $f(x) = 0$ is constant function.
(converges) \Rightarrow continuous function)

② For each $n \in \mathbb{N}$, choose $x = \frac{1}{n + \frac{1}{2}}$, then $f_n(x) = \sin^2(n\pi + \frac{\pi}{2}) = 1$ for any $n \in \mathbb{N}$.

Suppose $\{f_n(x)\}$ converges uniformly to $f(x) > 0$ for any $x \in \mathbb{R}$.

Given $\epsilon = \frac{1}{2} > 0$, $\exists N \in \mathbb{N}$ s.t. $|f_n(x) - 0| < \epsilon$ as $n \geq N$ for any $x \in \mathbb{R}$.

$$\Rightarrow \left| \sin^2\left(\frac{\pi}{n}\right) \right| < \frac{1}{2} \text{ as } n \geq N \text{ for any } x \in \mathbb{R}.$$

Choose $x = \frac{1}{N + \frac{1}{2}}$, then $\sin^2(N\pi + \frac{\pi}{2}) = 1 < \frac{1}{2}$. (contradiction)

So $\{f_n(x)\}$ does not uniformly converge on \mathbb{R} .

③ Now, since $\sin^2 \frac{\pi}{x} \geq 0$, $\frac{1}{n+1} \leq x \leq \frac{1}{n}$, then $\sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^{\infty} f_n(x)$ for all $x \in \mathbb{R}$.

By case 1 of ①, if $x \leq 0$ or $x \geq 1$, then $f_n(x) = 0 \quad \forall n \in \mathbb{N} \Rightarrow \sum_{n=1}^{\infty} f_n(x) = 0$ for all $x \leq 0$ or $x \geq 1$.

Given $0 < x < 1$, then $\frac{1}{n_0+1} \leq x \leq \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$. $\Rightarrow f_{n_0}(x) = \sin^2 \frac{\pi}{x}$.

But $f_n(x) = 0$ for all $n \in \mathbb{N}$, $n \neq n_0$. Then $\sum_{n=1}^{\infty} f_n(x) = \sin^2 \frac{\pi}{x}$.

Define $g(x) = \begin{cases} 0, & x \leq 0 \text{ or } x \geq 1 \\ \sin^2 \frac{\pi}{x}, & 0 < x < 1. \end{cases}$

Thus, $\sum_{n=1}^{\infty} |f_n(x)| = \sum_{n=1}^{\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$.

$\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergence.

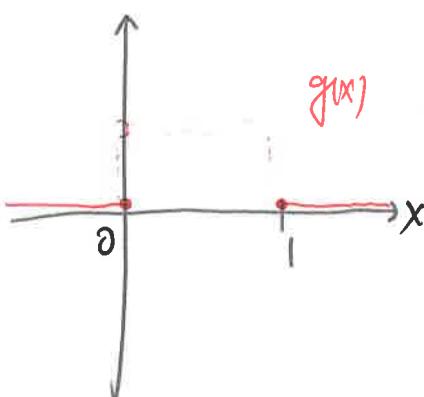
④ Claim: $\sum_{n=1}^{\infty} f_n(x)$ does not uniformly converge.

Suppose $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Since $\sum_{n=1}^{\infty} f_n(x) = g(x)$ for all $x \in \mathbb{R}$, then $g(x)$ is continuous on \mathbb{R} .

But $g(x)$ is not continuous at $x=0$. (contradiction)

So $\sum_{n=1}^{\infty} f_n(x)$ does not converge uniformly.



Rudin, ch 7, *6.

Prove that the series $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n + n}{n^2}$ converges uniformly in every bounded interval, but does not converge absolutely for any value of x .

(pf) Let $[a, b]$ be bounded interval.

① Claim: $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n}{n^2}$ converges uniformly and converges absolutely on $[a, b]$.

Let $M = \max\{|a|, |b|\}$.

$$\left| \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{M^n}{n^2} = M^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (\text{if } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges (By thm 7.10)})$$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n}{n^2}$ converges uniformly and absolutely on $[a, b]$.

② Now, we know that $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges to $\ln 2$.

③ Let $S_k(x) = \sum_{n=1}^k (-1)^n \cdot \frac{x^n + n}{n^2} = \sum_{n=1}^k (-1)^n \cdot \frac{x^n}{n^2} + \sum_{n=1}^k \frac{(-1)^n}{n}$ for all $k \in \mathbb{N}$.

Given $\epsilon > 0$, since $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n}{n^2}$ converges uniformly, then $\exists N_1 \in \mathbb{N}$ s.t.

if $k, l \geq N_1$, we have $\left| \sum_{n=1}^k (-1)^n \cdot \frac{x^n}{n^2} - \sum_{n=1}^l (-1)^n \cdot \frac{x^n}{n^2} \right| < \epsilon$ for any $k, l \geq N_1$, for any $x \in [a, b]$.

since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, then $\exists N_2 \in \mathbb{N}$ s.t.

if $k, l \geq N_2$, we have $\left| \sum_{n=1}^k \frac{(-1)^n}{n} - \sum_{n=1}^l \frac{(-1)^n}{n} \right| < \epsilon$ for any $k, l \geq N_2$.

Choose $N_3 = \max\{N_1, N_2\} \in \mathbb{N}$. If $k, l \geq N_3$, then we have

$$|S_k(x) - S_l(x)| \leq \left| \sum_{n=1}^k \frac{(-1)^n}{n^2} - \sum_{n=1}^l \frac{(-1)^n}{n^2} \right| + \left| \sum_{n=1}^k \frac{(-1)^n}{n} - \sum_{n=1}^l \frac{(-1)^n}{n} \right| < 2\epsilon$$

for any $k, l \geq N_3$ for any $x \in [a, b]$. Thus, $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^n + n}{n^2}$ converges uniformly on $[a, b]$.

④ $\sum_{n=1}^{\infty} |(-1)^n \cdot \frac{x^2+n}{n^2}| = \sum_{n=1}^{\infty} \frac{x^2+n}{n^2} \geq \sum_{n=1}^{\infty} \frac{1}{n}$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge,

then $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^2+n}{n^2}$ does not converge absolutely.



Rudin, ch7, *9. $f_n(x) = \frac{x}{1+nx^2}$

Show that $\{f_n\}$ converges uniformly to a function f , and that the equation

$f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$ is correct if $x \neq 0$, but false if $x=0$.

(pf)

$$\textcircled{1} \quad \because \frac{1+nx^2}{2} \geq \sqrt{nx^2} = \sqrt{n}|x| \quad \forall x \in \mathbb{R}, \text{ then } |f_n(x)| = \frac{|x|}{1+nx^2} \leq \frac{|x|}{2\sqrt{n}|x|} = \frac{1}{2\sqrt{n}} \quad \forall x \in \mathbb{R} \setminus \{0\}.$$

$$\text{If } \lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{x}{1+nx^2} = 0 \quad \text{for any } x \in \mathbb{R}, \text{ to define } f(x) \Rightarrow \forall x \in \mathbb{R},$$

$$\text{and let } M_n = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| = \sup_{x \in \mathbb{R}} |f_n(x)| \leq \frac{1}{2\sqrt{n}} \quad \text{for all } n \in \mathbb{N}, \text{ then}$$

since $\lim_{n \rightarrow \infty} M_n \leq \lim_{n \rightarrow \infty} \frac{1}{2\sqrt{n}} = 0$, by thm 7.9, we have $\{f_n\}$ converges uniformly to f .

(2)

$$\text{Now, } f'_n(x) = \frac{1+nx^2 - 2nx^2}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2} \quad \text{and. } f'(x) = 0 \quad \forall x \in \mathbb{R}.$$

$$\lim_{n \rightarrow \infty} f'_n(x) = \lim_{n \rightarrow \infty} \frac{1-nx^2}{(1+nx^2)^2} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n^2} - \frac{x^2}{n}}{\frac{1}{n^2} + \frac{2x^2}{n} + x^4} = \frac{0}{x^4} = 0 = f'(x) \\ \text{for any } x \in \mathbb{R} \setminus \{0\}.$$

As $x \neq 0$, then $f'_n(0) = 1$ for any $n \in \mathbb{N}$.

$$\text{Then } \lim_{n \rightarrow \infty} f'_n(0) = 1 \neq 0 = f'(0).$$

