Name and Student ID's: ____________________

Homework 9, Advanced Calculus 1

A topological space X is *sequentially compact* if every sequence has a convergent subsequence.

A compact space is sequentially compact (cf. Theorem 2.37 of Rudin, which is valid for general topological spaces). The first two problems below will prove that for metric space, the two compactness properties are actually equivalent.

1. Prove that for a sequentially compact metric space (X, d), every open cover has a *countable* subcover.

Solution: The statement follows from our earlier homework exercises. First we recall Exercises 23,24 of Chapter 2 (Problem 2, 3 of Homework 4). The two problems together say that if every infinite subset of a metric space (X, d) has a limit point, then it has a countable base. A sequentially compact metric space certain satisfies the sufficient condition, since every infinite subset contains a sequence, and the limit of its convergent subsequence is a limit point. Let $\mathcal{U} = \{U_i\}_{i=1}$ be a countable base: every open subset is a union of sets in $\mathcal{U}$.

Given a open cover $X = \cup_\alpha G_\alpha$ of X, for every α, we have $G_\alpha = \cup_{i_\alpha} U_{i_\alpha}$ where $U_{i_\alpha} \in \mathcal{U}$. The set $\{U_{i_\alpha}\}_{\alpha, i_\alpha}$ is a subset of $\mathcal{U}$ and therefore countable, which we re-label by $\{U_{i_j}\}_j \subset \{U_i\}_i$ and we have $X = \cup_\alpha G_\alpha = \cup_j U_{i_j}$. For each j, pick ONE G_α that contains U_{i_j}, call it G_j. The choice is possible due to the *axiom of choice*. We then clearly have $X = \cup_j G_j$, a countable subcover.

2. Use Problem 1 to show that (X, d) is compact.

Solution: Let $X = \cup_\alpha G_\alpha$. If there is a finite subcover, we are done. If not, by Problem 1, there is a countable subcover $X = \cup_{j=1}^\infty G_j$. Suppose this cover has NO finite subcover. Then we may take a sequence as follows. Let $x_1 \in G_1$, and inductively, for each k, take $x_k \notin \cup_{j=1}^k G_j$ possible due to the assumption that the open cover above has no finite subcover. Since X is sequentially compact, the sequence $\{x_k\}_k$ has a convergent subsequence, which we still call $\{x_k\}$ for convenience. Now $x_k \to x$ as $k \to \infty$ and $x \in G_n$ for some n. Since $x_k \to x$, there is $K \in \mathbb{N}$ so that $x_k \in G_n$ $\forall k > K$. Increasing K if necessary, we may assume that $K > n$. But then we have $x_k \in G_n$ for some $k > n$, contradicting our construction that $x_k \notin \cup_{j=1}^k G_j$. We conclude that the countable open cover above has a finite subcover and therefore X is compact.

3. Rudin Chapter 7 Exercise 8

Solution: Let $g_m(x) = \sum_{n=1}^m c_n I(x - x_n)$. We first show that $g_m \rightrightarrows f$ on $[a, b]$. Since $|c_n I(x - x_n)| \leq |c_n|$ $\forall x$ and $\sum |c_n|$ converges, this follows easily from Theorem 7.10 (the Weirstrass M-Test). The continuity of f then follows from continuity of each g_m, which follows from continuity of each $I(x - x_n)$ on $[a, b] \backslash \{x_j\}$.

For $x \in [a, b] \backslash \{x_j\}$, we have $x \neq x_n$. There exists $\delta > 0$ so that $(x - \delta, x + \delta) \cap \{x_j\} = \emptyset$. For all $y \in (x - \delta, x + \delta)$, we see that x_n is less or greater than BOTH x and y. Therefore $I_(x - x_n) = I(y - x_n)$ and the function is a constant on $(x - \delta, x + \delta)$ which is certainly continuous.

4. Rudin Chapter 7 Exercise 9

Solution: Since $f_n \rightrightarrows f$ and each f_n is continuous, so is f. For all $\epsilon > 0$, there exist $N \in \mathbb{N}$ and $\delta > 0$ so that

- $|f_n(t) - f(t)| < \frac{\epsilon}{2}$ for all $n > N$ and $t \in E$.
- $|f_(t) - f(x)| < \frac{\epsilon}{2}$ if $|t - x| < \delta$.

Increasing N if necessary, since $x_n \to x$, we have $d(x_n, x) < \delta$ for n large enough. We then have

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \epsilon$$

for all n satisfying the requirement above.

The converse is false. Consider $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}$ on $(0,1)$, which has a pointwise limit $f = 0$. We observe that $0 \leq f(x) \leq \frac{x^2}{(1-nx)^2}$. For every $x \in (0,1)$ and $x_n \to x$, there exists $\epsilon > 0$ so that $x_n \in (x-\epsilon, x+\epsilon) \subset (0,1)$ for all n large enough. On that neighborhood, we have

$$0 \leq f_n(x) \leq \frac{x^2}{[1-n(x+\epsilon)]^2},$$

and the right hand side approaches 0 as $n \to \infty$.

However, the convergence is not uniform since for every n, $f_n(\frac{1}{n}) = 1$ and therefore f_n does not converge uniformly to 0.

5. Rudin Chapter 7 Exercise 10

Solution: We first prove that f is discontinuous precisely on $\mathbb{Q}$, which is countable and dense. Clearly, every function (nx) is discontinuous at $\frac{p}{n}$ for $p \in \mathbb{N}$ with left limit $= 1$ and right limit $= 0$. Therefore, each $\frac{p}{q} \in \mathbb{Q}$, is a discontinuity for functions $\{\frac{(jqx)}{j^2q^2}\}_{j\in\mathbb{N}}$ in the series. All the other terms are continuous. Therefore $\lim_{x\to\frac{p}{q}^-} f(x) - \lim_{x\to\frac{p}{q}^+} f(x) = \sum_{j=1}^{\infty} \frac{1}{j^2q^2} > 0$. On the other hand, each $\frac{(nx)}{n^2}$ is continuous on $\mathbb{Q}^c$ and so is the partial sum $\sum_{n=1}^{m} \frac{(nx)}{n^2}$. Since $|\frac{(nx)}{n^2}| \leq \frac{1}{n^2}$ and $\sum_n \frac{1}{n^2}$ converge, $\sum_{n=1}^{m} \frac{(nx)}{n^2} \rightrightarrows f$ and therefore f is continuous on $\mathbb{Q}^c$. Therefore, f is discontinuous *precisely* on $\mathbb{Q}$.

For the Riemann integrability, note that each (nx) is discontinuous at $\{\frac{p}{n}\}_{p\in\mathbb{N}}$, and each bounded interval $[a,b]$ can only contain finitely many points of them, and so is the partial sum $\sum_{n=1}^{m} \frac{(nx)}{n^2}$. Having finitely many discontinuities, it follows that $\sum_{n=1}^{m} \frac{(nx)}{n^2}$ is Riemann integrable on $[a,b]$, and so is its uniform limit f as $m \to \infty$.

6. Rudin Chapter 7 Exercise 15

Solution: We claim that f satisfying these conditions is a constant function.

If not, there are points $x < y$ so that $\epsilon = |f(x) - f(y)| > 0$. Since $\{f_n(t)\}$ is equicontinuous, there exists $\delta > 0$ so that $|s-t| < \delta \Rightarrow |f_n(s) - f_n(t)| < \epsilon$ for all n. Pick n larger enough so that $(\frac{x}{n}, \frac{y}{n}) \subset (s,t)$. Then $|f_n(\frac{x}{n}) - f_n(\frac{y}{n})| < \epsilon$. But since $f_n(t) = f(nt)$, we have $|f_n(\frac{x}{n}) - f_n(\frac{y}{n})| = |f(x) - f(y)| = \epsilon$, a contradiction.

7. Rudin Chapter 7 Exercise 16

Solution: Since $f_n(x)$ is pointwise convergent, it is pointwise bounded. Therefore $\{f_n\}$ satisfies the condition for Arzela-Azcoli Theorem, and therefore it has a uniformly convergent subsequence. But since f_n has a pointwise limit, the subsequence is the sequence itself.

8. Rudin Chapter 7 Exercise 11

Solution: Repeat the proof of Theorem 3.42 and conclude that $\sum f_n g_n$ is uniformly Cauchy and therefore uniformly convergent.

9. Rudin Chapter 7 Exercise 12

Solution:

The solution is summarized from Group 4 - greatly appreciated!

First note that since $|f_n| \leq g$ and $f_n \rightrightarrows f$, $|f| \leq g$ as well. Since $\int_0^\infty g\ dx < \infty$, $\int_0^\infty f_n\ dx$ and $\int_0^\infty f_n\ dx$ exist and are all finite.

Second, $\int_0^\infty g\ dx < \infty$ means that the sequence $s_m := \int_{\frac{1}{m}}^m g\ dx$ converges to $s = \int_0^\infty g\ dx$. That is, $s - s_m \to 0$ as $m \to \infty$. Precisely, for all $\epsilon > 0$, there is $M_\epsilon \in \mathbb{N}$ so that

$$\int_{[\frac{1}{m},m]^c} g\ dx = \lim_{a\to 0} \int_a^{\frac{1}{m}} g\ dx + \lim_{b\to\infty} \int_m^b g\ dx < \frac{\epsilon}{3}\ \ \forall m > M_\epsilon. \tag{1}$$

Now we estimate

$$\left| \int_0^\infty f_n\ dx - \int_0^\infty f\ dx \right| \leq \int_{[\frac{1}{m},m]^c} |f_n| dx + \int_{\frac{1}{m}}^m |f_n - f|\ dx + \int_{[\frac{1}{m},m]^c} |f| dx.$$

Since $|f_n|, |f| \leq g$, we have

$$\left| \int_0^\infty f_n\ dx - \int_0^\infty f\ dx \right| \leq 2 \int_{[\frac{1}{m},m]^c} g\ dx + \int_{\frac{1}{m}}^m |f_n(x) - f(x)|\ dx. \tag{2}$$

Since $f_n \rightrightarrows f$ on every compact interval, there exists N_ϵ so that $\sup_{[\frac{1}{m},m]} |f_n(x) - f(x)| < \frac{\epsilon}{3}$ for all $n > N_\epsilon$. Then, for all $n > N_\epsilon$, pick $m > M_\epsilon$ as above, then the right hand side of (2) is less than ϵ and we are done.

10. Rudin Chapter 7 Exercise 13

Solution: In this problem, we use the fact that a monotonic function can have at most countably many discontinuities.

Since $\{f_n(x)\}$ are uniformly bounded, it is pointwise bounded and therefore $\{f_n(x)\}$ has a convergent subsequence at every $x \in \mathbb{Q} \cap [0,1]$. Since this set is countable, there is a subsequence $\{f_{n_i}\}_i$ of $\{f_n\}$ that converges at every $r \in \mathbb{Q} \cap [0,1]$ to, say $f(r)$. We extend the domain of f to the entire $[0,1]$ by

$$f(x) = \sup_{r \leq x, rational} f(r).$$

We check that f is monotonically increasing. It is an increasing function on $\mathbb{Q} \cap [0,1]$. Indeed, take two rationals $r < s$. $f_{n_i}(r) \leq f_{n_i}(s)$ for all i and so are their limits as $i \to \infty$. For any $x < y$, take a rational number $r \in (x, y)$. Then $f(r) \geq f(r')$ for all rational numbers $r' \leq x$. Therefore, $f(y) \geq f(r) \geq \sup_{r' \in \mathbb{Q} \leq x} f(r') = f(x)$.

Next we prove that $f_{n_i}(x) \to f(x)$ as $i \to \infty$ for all x at which f is continuous. Let $\epsilon > 0$, there exists $\delta > 0$ so that $|t - x| < \delta \Rightarrow |f(t) - f(x)| < \frac{\epsilon}{2}$. Take two rational numbers $r, s \in (x - \delta, x + \delta)$ so that $r < x < s$. Monotonicity of g_i implies that

$$f_{n_i}(r) \leq f_{n_i}(x) \leq f_{n_i}(s).$$

Since $f_{n_i}(r), f_{n_i}(s) \to f(r), f(s)$ respectively, there is $I \in \mathbb{N}$ so that $f_{n_i}(s) \leq f(s) + \frac{\epsilon}{2}$ and $f_{n_i}(r) \leq f(r) + \frac{\epsilon}{2}$ for all $i > I$. Furthermore, since $r, s \in (x - \delta, x + \delta)$, continuity of f implies that $f(s) \leq f(x) + \frac{\epsilon}{2}$ and $f(r) \geq f(x) - \frac{\epsilon}{2}$. Combining all the estimates, we have

$$f(x) - \epsilon \leq f_{n_i}(x) \leq f(x) + \epsilon$$

for all $i > I$. Therefore, $f_{n_i}(x) \to f(x)$.

Finally, since $f(x)$ is monotonic, there are at most countably many points of discontinuities. The functions f_{n_i} converge at each of those point, and therefore we may take a further subsequence $\{f_{n'_i}\}$, which converges at every point of discontinuity. The subsequence of course still converge at point of continuity of f, and the proof is completed.

11. Prove Theorem 7.17, with additional assumption that f'_n is continuous for all n.

Solution: Since each $f'_n(x)$ is continuous, it is integrable and by the Fundamental Theorem of Calculus, we have

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f'_n(t)\, dt.$$

Then for all $x \in [0,1]$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + \int_{x_0}^{x} |f'_n(t) - f'_m(t)|\, dt.$$

Take $\epsilon > 0$, since f_n converge uniformly, it is uniformly Cauchy and there is $N_1 \in \mathbb{N}$ so that $|f'_n(t) - f'_m(t)| < \frac{\epsilon}{2}$ for all $t \in [0,1]$ and $n, m > N_1$. Since $\{f_n(x_0)\}$ converges, there is $N_2 \in \mathbb{N}$ so that $|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$ for all $n, m > N_2$. Take $N = max(N_1, N_2)$, then $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m > N$, $x \in [0,1]$ and therefore $\{f_n\}$ is uniformly Cauchy and convergent.

Consider, for $t \neq x$,

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x} = \frac{\int_x^t f'_n(s)\, ds}{t - x}.$$

Since f'_n uniformly converge to, say, g, we have

$$\phi_n(t) \to \frac{\int_x^t g(s)\, ds}{t-x}$$

as $n \to \infty$. On the other hand, since $f_n \rightrightarrows f$, we have $\phi_n(t) \to \frac{f(t)-f(x)}{t-x}$ as $n \to \infty$ and therefore

$$\frac{f(t)-f(x)}{t-x} = \frac{\int_x^t g(s)\, ds}{t-x}.$$

Let $G(t) = \int_x^t g(s)\, ds$. Then $G(x) = 0$ and we have, by the Fundamental Theorem of Calculus,

$$\lim_{t\to x} \frac{\int_x^t g(s)\, ds}{t-x} = \lim_{t\to x} \frac{G(t)-G(x)}{t-x} = G'(x) = g(x),$$

and therefore we have

$$\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = g(x),$$

The left hand side is precisely the definition for $f'(x)$.