Final Exam
Undergraduate Analysis

Department of Mathematics
National Cheng-Kung University

January, 9 2018 (Fall)

- You must work alone.
- Any answer you produce must be supported with sufficient reasons. A correct answer without any reason will receive very little points.
- Turn in to office 203 before 17:30 on Friday, January 11, 2018.
- Good Luck!

1. Given a measure space $(X, \mathcal{M}, \mu)$ such that $\mu(X) < \infty$, and numbers $1 \leq p < q \leq \infty$, prove that

$$\mathcal{L}^q(\mu) \subset \mathcal{L}^p(\mu).$$

Note: $q = \infty$ is considered here.

2. Let f be an even function on $[-\pi, \pi]$ such that $f(x) = \cos(2x)$ on $[0, \frac{\pi}{2}]$ and $f(x) = 1$ on $(\frac{\pi}{2}, \pi]$. Find its Fourier series.

3. Use results introduced in class to evaluate

$$\int_{\mathbb{R}} \frac{\sin^2(3x)}{4x^2}\, dx.$$

4. Determine $f(t) \in \mathcal{L}^1$ (explicitly) so that

$$\int_{\mathbb{R}} f(t-y)e^{-\frac{y^2}{2}}\, dy = e^{-\frac{t^2}{4}}$$

for all t.

5. Prove that Heisenberg's inequality for the case $a = \alpha = 0$ is an equality if and only if

$$f(x) = Ke^{-\frac{cx^2}{2}}$$

for some $K, c \in \mathbb{R}$ with $c > 0$.

6. Solve for $\mathcal{L}^1$ function $u(x,t)$ on $\mathbb{R}^2$ satisfying

$$\begin{cases} u_t = iu_{xx} \\ u(0,t) = f(t). \end{cases}$$

for some $f \in \mathcal{L}^1$. Express your final answer in terms of functions given above and other explicit functions *without* $\widehat{\ }$ sign. Integration sign is allowed (i.e. you don't have to explicitly do the integration).

Assume that the integral

$$\int_{\mathbb{R}} u(x,t)e^{-it\tau}\, dt$$

is bounded for all $x, \tau \in \mathbb{R}$ and is 0 for all $x \in \mathbb{R}$ and $\tau \leq 0$.