HW1 Solution

1. Equivalence relation

① reflexive: $\forall x \in [0,1)$, $x \sim x$ by $x - x = 0 \in \mathbb{Q}$.

② symmetric: If $x \sim y$, then $\exists q \in \mathbb{Q}$ s.t. $x - y = q$.
For such (x, y), we thus have $y - x = -q \in \mathbb{Q}$
$\Rightarrow y \sim x$.

③ translation: If $x \sim y$, $y \sim z$, then $\exists q_1, q_2 \in \mathbb{Q}$
s.t. $x - y = q_1$ & $y - z = q_2$.
For such (x, y), (y, z), we thus have
$x - z = (x - y) + (y - z) = q_1 + q_2 \in \mathbb{Q}$
$\Rightarrow x \sim z$.

partition of $[0,1)$ into translations of N

Pick any $x \in [0,1)$, $\exists y \in N$ s.t. $x \in [y]$.
Then $\exists q \in \mathbb{Q}$ s.t. $x - y = q \Rightarrow x = y + q$.

Observe: $0 \le x, y < 1 \Rightarrow -1 < x - y < 1 \Rightarrow q \in (-1, 1) \cap \mathbb{Q}$.

- If $q \in (-1, 0)$, then $q + 1 \in (0,1) \Rightarrow \exists r \in R \cap (0,1)$
s.t. $q + 1 = r$.
Since $y = x - q = x - (r-1) = x + 1 - r \in [1-r, 2)$
and $y \in [0,1)$ by def. of y $\Rightarrow y \in [1-r, 1) = [1-r, 2) \cap [0,1)$.
In this case, $x = y + q = y + (r-1) \in N_r$.

- If $q \in [0,1)$, then $y = x - q \in [-q, 1-q)$ and by def of y,
$y \in [0,1) \Rightarrow y \in [0, 1-q)$.
In this case, $x = y + q \in N_q$.

$\therefore x \in \bigcup_{r \in R} N_r$. Since x is arbitrarily, we get $[0,1) \subseteq \bigcup_{r \in R} N_r$.

2. let $\mathcal{A}$ be an Index set, $\{\mathcal{M}_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection of σ-algebras.

By def. of σ-algebra, we need to verify $\bigcap_{\alpha\in\mathcal{A}} \mathcal{M}_\alpha := \mathcal{M}$ satisfy

① If $E\in\mathcal{M}$, then $E^c\in\mathcal{M}$.

② If $E, F\in\mathcal{M}$, then $E\cup F\in\mathcal{M}$

③ If $\{E_i\}_{i=1}^{\infty}\subseteq\mathcal{M}$, then $\bigcup_{i=1}^{\infty} E_i\in\mathcal{M}$.

① If $E\in\mathcal{M}$, then $E\in\mathcal{M}_\alpha$ $\forall\alpha\in\mathcal{A}$, thus $E^c\in\mathcal{M}_\alpha$ $\forall\alpha\in\mathcal{A}$, thus $E^c\in\bigcap_{\alpha\in\mathcal{A}}\mathcal{M}_\alpha=\mathcal{M}$.

② If $E, F\in\mathcal{M}$, then $E, F\in\mathcal{M}_\alpha$ $\forall\alpha\in\mathcal{A}$

$$\Rightarrow E\cup F\in\mathcal{M}_\alpha \quad \forall\alpha\in\mathcal{A}$$

$$\Rightarrow E\cup F\in\mathcal{M}=\bigcap_{\alpha\in\mathcal{A}}\mathcal{M}_\alpha.$$

③ If $\{E_i\}_{i=1}^{\infty}\subseteq\mathcal{M}\Rightarrow\{E_i\}_{i=1}^{\infty}\subseteq\mathcal{M}_\alpha^{\forall\alpha}\Rightarrow\bigcup_{i=1}^{\infty}E_i\in\mathcal{M}_\alpha \ \forall\alpha$

$$\Rightarrow\bigcup_{i=1}^{\infty}E_i\in\mathcal{M} \quad ※$$

3. $g(x)=\begin{cases}0 & 0\le x\le \frac{1}{2}\\ 1 & \frac{1}{2}<x\le 1\end{cases}$

$f_{2k}(x)=\begin{cases}0 & 0\le x\le \frac{1}{2}\\ 1 & \frac{1}{2}<x\le 1\end{cases}$

$f_{2k+1}(x)=\begin{cases}1 & 0\le x\le \frac{1}{2}\\ 0 & \frac{1}{2}<x\le 1\end{cases}$

- For $x\in[0,\frac{1}{2}]$, we have $f_{2k}(x)=0 \ \forall k$;
  for $x\in(\frac{1}{2},1]$, we have $f_{2k+1}(x)=0 \ \forall k$

  $\Rightarrow \liminf_{n\to\infty} f_n(x)=0.$

- $\forall n\in\mathbb{N}$, $\int_0^1 f_n(x)\,dx = \int_0^{\frac{1}{2}} f_n(x)\,dx+\int_{\frac{1}{2}}^1 f_n(x)\,dx$

$$=\begin{cases}\int_0^{\frac{1}{2}} 0\,dx+\int_{\frac{1}{2}}^1 1\,dx=\frac{1}{2} & \text{if } n \text{ even.}\\ \int_0^{\frac{1}{2}} 1\,dx+\int_{\frac{1}{2}}^1 0\,dx=\frac{1}{2} & \text{if } n \text{ odd.}\end{cases}$$

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4. $f_n(x)=\begin{cases}\frac{1}{n}, & |x|\le n\\ 0, & |x|>n\end{cases}$

($f_n\to f=0$ unif.)

For $x\in[-n,n]$, $f_n(x)=\frac{1}{n}\Rightarrow \lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}\frac{1}{n}=0.$

For $x\in(-\infty,-n)\cup(n,\infty)$, $f_n(x)=0\Rightarrow \lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} 0=0.$

$\therefore$ $f_n\to f:=0$ pointwisely.

Moreover, $\sup_{x\in\mathbb{R}}|f_n(x)-f(x)|=\sup_{x\in\mathbb{R}}|f_n(x)-0|$

$=\sup_{x\in\mathbb{R}}|f_n(x)|=\frac{1}{n}\to 0$ as $n\to\infty$.

$\therefore$ $f_n\to f$ uniformly.

($\int_{-\infty}^{\infty} f_n\,dx=?$)

$$\int_{-\infty}^{\infty} f_n\,dx=\int_{-n}^{n}\frac{1}{n}\,dx=\frac{1}{n}\cdot 2n=2 \quad ※$$

5. "$\Rightarrow$" (If $\mathcal{R}$ is σ-algebra)

- For $E, F \in \mathcal{R}$, since $F^c \in \mathcal{R} \Rightarrow E \cap F^c \in \mathcal{R}$.
- For $E_1, E_2, \cdots, E_n \in \mathcal{R} \Rightarrow \bigcup_{i=1}^{n} E_i \in \mathcal{R}$ is obviously.
- For $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{R} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{R}$ is obviously by def.

"$\Leftarrow$" (If $X \in \mathcal{R}$)

- If $E \in \mathcal{R}$, since $X \in \mathcal{R}$, $X - E = E^c \in \mathcal{R}$.
- If $E, F \in \mathcal{R}$, then $E \cup F \in \mathcal{R}$ by def. of σ-ring.
- If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{R} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{R}$ holds by def of σ-ring. $\divideontimes$

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6. (a) Suppose X is the universal set.

Pick $\mathcal{C} = \{E_k\}_{k=1}^{\infty} \subseteq \mathcal{M}$ for all $E_k \neq \phi$, and if $E_k \in \mathcal{C}$, $E_k^c \in \mathcal{C}$.

For any $x \in X$, we may define

$$C_x = \bigcap_{\substack{j \in \mathbb{N} \\ x \in E_j}} E_j$$

Then $C_x \in \mathcal{M}$ $\forall x$.

<u>claim 1: C_x is non-empty</u>

This is trivial by $x \in C_x$.

<u>claim 2: If $y \in C_x$, then $C_x = C_y$</u>

$\boxed{\text{"}\supseteq\text{"}}$ is obvious.

$\boxed{\text{"}\subseteq\text{"}}$ If E_k contains y, then x must be in E_k, or will lead to a contradiction because if $x \in E_k^c$, then $y \in C_x \subseteq E_k^c$ but $y \in E_k$, $\divideontimes$.

$\therefore x \in C_y \Rightarrow C_x \subseteq C_y$.

claim 3: $\forall C_x \in \{C_x\}_{x\in X}$, $C_x \cap C_y = \phi$ if $C_x \neq C_y$

If $\exists z \in C_x \cap C_y$, then $C_x = C_z$ & $C_z = C_y \Rightarrow C_x = C_y$, $\divideontimes$.
$\therefore$ $C_x \cap C_y = \phi$ $\forall$ C_x, C_y with $C_x \neq C_y$.

claim 4: $\{C_x\}_{x\in X}$ with infinitely many elements.

Define $U: P(\{C_x\}_{x\in X}) \longrightarrow P(X)$ by

$$\{C_x\}_{x\in E\subset X} \longmapsto \bigcup_{x\in E\subset X} C_x$$

Then since for any $E_k \in \mathcal{C}$, $E_k = \bigcup_{x\in E} C_x$,
$\Rightarrow E_k \in P(X)$ & thus $\mathcal{C} \subseteq P(X) \Rightarrow P(\{C_x\}_{x\in X})$ is infinite.

Thus $\{C_x\}_{x\in X}$ is infinite.

So for any $\{A_k\}_{k\in\mathbb{N}} \subseteq \{C_x\}_{x\in X}$ is any sequence which we want to find.

(b) By (a), $\exists \{A_k\}_{k\in\mathbb{N}}$ s.t. $A_k \neq \phi$ $\forall k$ & $A_k \cap A_n = \phi$ $\forall k \neq n$.

Define $U: P(\{A_k\}_{k\in\mathbb{N}}) \longrightarrow M$ by

$$\{A_{k_1}, A_{k_2}, \cdots, A_{k_r}\} \mapsto \bigcup_{j=1}^{r} A_{k_j}$$

Then $\bigcup_{j=1}^{r} A_{k_j} \neq \bigcup_{j=1}^{\ell} A_{\tilde{k}_j}$ $\forall$ $\{A_{k_1}, A_{k_2}, \cdots, A_{k_r}\} \neq \{A_{\tilde{k}_1}, A_{\tilde{k}_2}, \cdots, A_{\tilde{k}_\ell}\}$ (+)
by $A_k \cap A_n = \phi$ $\forall k \neq n$.

Since $|P(\{A_k\}_{k\in\mathbb{N}})|$ is uncountable & $\bigcup_{j=1}^{r} A_{k_j} \in M$ for all $\{A_{k_1}, A_{k_2}, \cdots, A_{k_r}\} \in P(\{A_k\}_{k\in\mathbb{N}})$ with (+) holds, we get M is uncountable. $\divideontimes$

7. We only need to show If $\{E_K\}_{K=1}^{\infty} \subseteq \mathcal{A}$, then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.

Define $F_K = \bigcup_{J=1}^{K} E_J$, then $F_K \in \mathcal{A}$ and $F_K \subset F_{K+1}$ $\forall K$.

By property of $\mathcal{A}$ which have, we get $\bigcup_{K=1}^{\infty} F_K \in \mathcal{A}$.

This means that $\bigcup_{i=1}^{\infty} E_i = \bigcup_{K=1}^{\infty} F_K \in \mathcal{A}$, done.