Name and Student ID's: ____

Homework 2, Undergraduate Analysis

Refer to class notes for definitions of sets and functions in this homework assignment.

1. Prove that $\mathcal{E}$, the collection of elementary sets in $\mathbb{R}^n$, is *not* closed under countable union.

2. Verity the inequalities for the "distance function" $d(A, B) = \mu^*(S(A, B))$ defined in class.

3. Given $A \in \mathcal{M}(\mu)$, and $\epsilon > 0$, prove that there exists closed set $F \in \mathcal{M}(\mathcal{E})$, open set $G \in \mathcal{M}(\mathcal{E})$, such that $F \subset A \subset G$ and
   - $\mu(G) \leq \mu(A) + \epsilon$
   - $\mu(F) \geq \mu(A) - \epsilon$.

4. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be monotonic.
   (a) Prove that left or right limit always exists at every point $x \in \mathbb{R}$.
   (b) Prove that the set function

$$\begin{aligned} \mu([a,b]) &= \alpha(b^+) - \alpha(a^+), \\ \mu([a,b)) &= \alpha(b^-) - \alpha(a^+), \\ \mu((a,b]) &= \alpha(b^+) - \alpha(a^-), \\ \mu((a,b)) &= \alpha(b^-) - \alpha(a^-) \end{aligned}$$

   is regular. Here, $\alpha(x^-) = \lim_{t \to x^-} \alpha(t)$ and $\alpha(x^+) = \lim_{t \to x^+} \alpha(t)$.

5. Rudin Chapter 11, Exercise 3.

6. Rudin Chapter 11, Exercise 15.

7. Given a measure space $(X, \mathcal{M}, \mu)$ and $f : X \to \bar{\mathbb{R}}$ such that $f^{-1}((r, \infty]) \in \mathcal{M}$ for all $r \in \mathbb{Q}$, prove that f is measurable.

8. Given a measure space $(X, \mathcal{M}, \mu)$ and $X = A \bigcup B$ with $A, B \in \mathcal{M}$, prove that a function f on X is measurable if and only if $f|_A$ and $f|_B$ are measurable on A and B, respectively.