HW 2 Solution

1. let $E=\{q_k\}_{k\in\mathbb{N}} \subseteq [0,1]\cap\mathbb{Q}$, with $q_k < q_{k+1}$, $\forall k\in\mathbb{N}$. Set $I_k=(q_k, q_{k+1})$, then I_k is elementary set, $\forall k$. But $I=\bigcup_{k=1}^{\infty} I_k$ is not a elementary set for I is union of infinitely many disjoint intervals.

2. $S(A,B)=(A-B)\cup(B-A)$, $d(A,B)=\mu^*(S(A,B))$, $A,B\subseteq\mathbb{R}^p$.

① $d(A,B)\le d(A,C)+d(C,B)$

⟨sol⟩
$$\begin{aligned}
d(A,B)&=\mu^*(S(A,B))=\mu^*((A-B)\cup(B-A))\\
&=\mu^*((A-C)\cup(C-B)\cup(B-C)\cup(C-A))\\
&=\mu^*((A-C)\cup(C-A)\cup(C-B)\cup(B-C))\\
&\le \mu^*((A-C)\cup(C-A))+\mu^*((C-B)\cup(B-C))\\
&=\mu^*(S(A,C))+\mu^*(S(C,B))=d(A,C)+d(C,B).
\end{aligned}$$

② $$\left.\begin{aligned} &d(A_1\cup A_2, B_1\cup B_2)\ (\mathrm{I})\\ &d(A_1\cap A_2, B_1\cap B_2)\ (\mathrm{II})\\ &d(A_1-A_2, B_1-B_2)\ (\mathrm{III})\end{aligned}\right\} \le \underset{\mu^*(S(A_1,B_1))}{d(A_1,B_1)}+\underset{\mu^*(S(A_2,B_2))}{d(A_2,B_2)}$$

⟨sol⟩ (I)
$$\begin{aligned}
d(A_1\cup A_2, B_1\cup B_2)&=\mu^*(S(A_1\cup A_2, B_1\cup B_2))\\
&=\mu^*([(A_1\cup A_2)\cap(B_1^c\cap B_2^c)]\cup[(B_1\cup B_2)\cap(A_1^c\cap A_2^c)])\\
&=\mu^*((A_1-B_1-B_2)\cup(A_2-B_1-B_2)\cup(B_1-A_1-A_2)\cup(B_2-A_1-A_2))\\
&\le\mu^*((A_1-B_1)\cup(A_2-B_2)\cup(B_1-A_1)\cup(B_2-A_2))\\
&=\mu^*((A_1-B_1)\cup(B_1-A_1)\cup(A_2-B_2)\cup(B_2-A_2))\\
&\le\mu^*((A_1-B_1)\cup(B_1-A_1))+\mu^*((A_2-B_2)\cup(B_2-A_2))=d(A_1,B_1)+d(A_2,B_2)
\end{aligned}$$

(II) $d(A_1 \cap A_2, B_1 \cap B_2)$

$$= \mu^*(S(A_1 \cap A_2, B_1 \cap B_2))$$

$$= \mu^*([(A_1 \cap A_2) - (B_1 \cap B_2)] \cup [(B_1 \cap B_2) - (A_1 \cap A_2)])$$

$$= \mu^*([(B_1^c \cap (A_1 \cap A_2)) \cup (B_2^c \cap (A_1 \cap A_2))] \cup [(A_1^c \cap (B_1 \cap B_2)) \cup (A_2^c \cap (B_1 \cap B_2))])$$

$$\leq \mu^*((A_1 - B_1) \cup (A_2 - B_2) \cup (B_1 - A_1) \cup (B_2 - A_2))$$

$$\leq \mu^*((A_1 - B_1) \cup (B_1 - A_1)) + \mu^*((A_2 - B_2) \cup (B_2 - A_2))$$

$$= d(A_1, B_1) + d(A_2, B_2)$$

(III) $d(A_1 - A_2, B_1 - B_2)$ $\qquad (B_1 \cap B_2^c)^c = B_1^c \cup B_2$

$$= \mu^*(S(A_1 - A_2, B_1 - B_2))$$

$$= \mu^*([(A_1 - A_2) - (B_1 - B_2)] \cup [(B_1 - B_2) - (A_1 - A_2)])$$

$$= \mu^*([(B_1^c \cap A_1 \cap A_2^c) \cup (B_2 \cap A_1 \cap A_2^c)] \cup [(A_1^c \cap B_1 \cap B_2^c) \cup (A_2 \cap B_1 \cap B_2^c)])$$

$$\leq \mu^*([(A_1 - B_1) \cup (B_2 - A_2) \cup (B_1 - A_1) \cup (A_2 - B_2)])$$

$$\leq \mu^*((A_1 - B_1) \cup (B_1 - A_1)) + \mu^*((A_2 - B_2) \cup (B_2 - A_2))$$

$$= d(A_1, B_1) + d(A_2, B_2).$$

③ $|\mu^*(A) - \mu^*(B)| \leq d(A, B)$

<sol> By ①, we thus have $d(A, \phi) \leq d(A, B) + d(B, \phi)$,

and $d(A, \phi) = \mu^*((A - \phi) \cup (\phi - A)) = \mu^*(A)$,

$d(B, \phi) = \mu^*((B - \phi) \cup (\phi - B)) = \mu^*(B)$, so we have

$\mu^*(A) \leq d(A, B) + \mu^*(B) \Rightarrow \mu^*(A) - \mu^*(B) \leq d(A, B)$.

Change A & B, we have $\mu^*(B) - \mu^*(A) \leq d(B, A) \overset{\shortparallel}{} d(A, B)$,

thus ③ get.

3. on $\mathcal{M}(\mu)$, $\mu^* = \mu$, so $\forall A \in \mathcal{M}(\mu)$,
$\mu(A) = \mu^*(A) = \inf \sum_{i=1}^{\infty} \mu(E_i)$, for $A \subseteq \bigcup_{i=1}^{\infty} E_i$,
and E_i: open set.

So given any $\varepsilon > 0$, we may pick $G = \bigcup_{i=1}^{\infty} \tilde{E_i}$
s.t. $A \subseteq \bigcup_{i=1}^{\infty} \tilde{E_i} = G$ and thus
$\mu(A) + \varepsilon = \inf \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon \geq \sum_{i=1}^{\infty} \mu(\tilde{E_i}) \geq \mu(G)$.
Since this is true $\forall A \in \mathcal{M}(\mu)$, thus
$A^c \in \mathcal{M}(\mu)$ also holds, so there exists
G s.t. $A^c \subseteq G$: open with

$$\mu(A^c) + \varepsilon \geq \mu(G)$$

又 $A \supseteq G^c$, 取 $F = G^c$, then since

$\mu(A) + \mu(A^c) = \mu(X) = \mu(G) + \mu(G^c)$
$\mu(A^c) + \varepsilon \geq \mu(G) = \mu(A) + \mu(A^c) - \mu(F)$
$\Rightarrow \mu(F) + \varepsilon \geq \mu(A)$ ※

4.

(a) WLOG, suppose $\alpha \nearrow$.

Fix any $a \in \mathbb{R}$, let $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ with $x_n \downarrow a$ arbitrary.
then since $\alpha \nearrow$, $\{\alpha(x_n)\}_{n\in\mathbb{N}}$ is decreasing and with $\alpha(a) \leq \alpha(x_n)$ $\forall n \in \mathbb{N}$.

$\Rightarrow \exists A \in \mathbb{R}$ s.t. $\lim_{n\to\infty} \alpha(x_n) = A$.

If $\exists \{x_n\}_{n\in\mathbb{N}}$ & $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ with $x_n \downarrow a$ & $y_n \downarrow a$, say $A = \lim_{n\to\infty} \alpha(x_n)$ & $B = \lim_{n\to\infty} \alpha(y_n)$.
Now let $E = \{x_n\}_{n\in\mathbb{N}} \cup \{y_n\}_{n\in\mathbb{N}}$ and rearrange it into another decreasing sequence on $\mathbb{R}$, say $\{z_n\}$, then since $a \leq z_n$ $\forall n$ and $x_n, y_n \downarrow a \Rightarrow z_n \downarrow a$.
Also, since $\alpha(a) \leq \alpha(z_n)$ $\forall n$, $\{\alpha(z_n)\}_{n\in\mathbb{N}}$ converges, say to C.
But $\{\alpha(x_n)\} \subseteq \{\alpha(z_n)\}$ with $\lim_{n\to\infty} \alpha(x_n) = A$ and
$\{\alpha(y_n)\} \subseteq \{\alpha(z_n)\}$ with $\lim_{n\to\infty} \alpha(y_n) = B$,
By uniqueness of limit will forces to $A = C = B$.
Thus we show the right limit always exists.
For left limit, the process is similar, so we done our work.

(b) First of all, we discuss the cases for $[a,b)$, $[a,b]$, $(a,b]$, and (a,b).

Since $\lim_{t\to a^\pm} \alpha(t) = \alpha(a\pm)$ and $\lim_{t\to b^\pm} \alpha(t) = \alpha(b\pm)$ exists, given $\varepsilon > 0$, $\exists \delta > 0$ s.t.

$a-\delta \quad a \quad a+\delta$

- If $a-\delta<t<a \Rightarrow |\alpha(t)-\alpha(a-)|<\frac{\varepsilon}{2}$
- If $a<t<a+\delta \Rightarrow |\alpha(t)-\alpha(a+)|<\frac{\varepsilon}{2}$
- If $b-\delta<t<b \Rightarrow |\alpha(t)-\alpha(b-)|<\frac{\varepsilon}{2}$
- If $b<t<b+\delta \Rightarrow |\alpha(t)-\alpha(b+)|<\frac{\varepsilon}{2}$. $\quad \alpha(b-)-\frac{\varepsilon}{2}<\alpha(t)<\alpha(b-)+\frac{\varepsilon}{2}$

For such δ,

$\boxed{[a,b)}$ $\quad [a, b-\frac{\delta}{2}] \subset [a,b) \subset (a-\frac{\delta}{2}, b)$

$$\Rightarrow \mu([a,b-\tfrac{\delta}{2}]) = \alpha((b-\tfrac{\delta}{2})+)-\alpha(a-) > \alpha(b-)-a(-)-\tfrac{\varepsilon}{2} \geq \mu([a,b))-\varepsilon.$$

$$\mu((a-\tfrac{\delta}{2},b)) = \alpha(b-)-\alpha((a-\tfrac{\delta}{2})+) < \alpha(b-)-\alpha(a-)+\tfrac{\varepsilon}{2} \leq \mu([a,b))+\varepsilon.$$

$\boxed{[a,b]}$ $\quad [a,b] \subseteq [a,b] \subset (a-\frac{\delta}{2}, b+\frac{\delta}{2})$.

$$\Rightarrow \mu([a,b]) \geq \mu([a,b])-\varepsilon$$

$$\mu((a-\tfrac{\delta}{2}, b+\tfrac{\delta}{2})) = \alpha((b+\tfrac{\delta}{2})-)-\alpha((a-\tfrac{\delta}{2})+)$$
$$< \alpha(b+)+\tfrac{\varepsilon}{2}-\alpha(a-)+\tfrac{\varepsilon}{2} = \mu([a,b])+\varepsilon$$

$\boxed{(a,b]}$ $\quad [a-\frac{\delta}{2}, b] \subset (a,b] \subset (a, b+\frac{\delta}{2})$

$$\Rightarrow \mu([a-\tfrac{\delta}{2},b]) = \alpha(b+)-\alpha((a-\tfrac{\delta}{2})-) > \alpha(b+)-\alpha(a-)-\tfrac{\varepsilon}{2}$$
$$> \mu((a,b])-\varepsilon$$

$$\mu((a,b+\tfrac{\delta}{2})) = \alpha((b+\tfrac{\delta}{2})-)-\alpha(a+)$$
$$< \alpha(b+)+\tfrac{\varepsilon}{2}-\alpha(a+) < \mu((a,b])+\varepsilon$$

$\boxed{(a,b)}$ $\quad [a+\frac{\delta}{2}, b-\frac{\delta}{2}] \subset (a,b) \subseteq (a,b)$.

$$\Rightarrow \mu([a+\tfrac{\delta}{2},b-\tfrac{\delta}{2}]) = \alpha((b-\tfrac{\delta}{2})+)-\alpha((a+\tfrac{\delta}{2})-) > \alpha(b-)-\alpha(a+)-\tfrac{\varepsilon}{2}$$
$$> \mu((a,b))-\varepsilon$$

$$\mu((a,b)) \leq \mu(a,b)+\varepsilon.$$

Thus μ is "regular" on these four cases.

Now consider the general case, say $A \in \mathcal{E}$, and $A = \bigcup_{i=1}^{n} I_i$, for some n & $I_i \cap I_j = \phi$ if $i \neq j$.

Then given $\varepsilon > 0$, for each I_i we may find I_i^{o} & I_i^{x} with $I_i^{x} \underset{\text{closed}}{\subseteq} I_i \subseteq I_i^{o}$ s.t. $\mu(I_i^{o}) - \frac{\varepsilon}{n} \leq \mu(I_i) \leq \mu(I_i^{x}) + \frac{\varepsilon}{n}$,

Then $I^{x} := \bigcup_{i=1}^{n} I_i^{x} \underset{\text{closed}}{\subseteq} A \underset{\text{open}}{\subseteq} \bigcup_{i=1}^{n} I_i^{o} =: I^{o}$ will with

$$\mu(I^{o}) = \mu\left(\bigcup_{i=1}^{n} I_i^{o}\right) \leq \sum_{i=1}^{n} \mu(I_i^{o}) \leq \sum_{i=1}^{n}\left(\mu(I_i) + \frac{\varepsilon}{n}\right)$$
$$= \mu(A) + \varepsilon$$

and

$$\mu(I^{x}) = \mu\left(\bigcup_{i=1}^{n} I_i^{x}\right) = \sum_{i=1}^{n} \mu(I_i^{x}) \geq \sum_{i=1}^{n}\left(\mu(I_i) - \frac{\varepsilon}{n}\right)$$
$$= \mu(A) - \varepsilon.$$

So μ is regular on elementary sets.

5. Set $E := \{x \mid \{f_n(x)\} \text{ converges}\}$.

Fix any $x \in \mathbb{R}$, then $\{f_n(x)\}$ is a "sequence" on $\mathbb{R}$, so $\{f_n(x)\}$ converges

$\Leftrightarrow \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$.

Set $g = \limsup_{n\to\infty} f_n$ & $h = \liminf_{n\to\infty} f_n$, then both g, h are measurable function $\Rightarrow g-h$ also meas.

$\Rightarrow E = \{x \mid g-h \geq 0\} \cap \{x \mid g-h \leq 0\}$ thus measurable.

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6. $\mathcal{R}$: ring of all elementary subsets of $(0,1]$.

$\phi([a,b]) := \phi([a,b)) = \phi((a,b]) = \phi((a,b)) = b-a$ $\forall a,b$ with $0 < a \leq b \leq 1$

$\phi((0,b)) := \phi((0,b]) = 1+b$

- Additive: For any $A, B \in \mathcal{R}$ with $A \cap B = \phi$, say $A = \bigcup_{i=1}^{k} I_i$, $B = \bigcup_{j=1}^{m} I_j$, for some k, m & I_l be elementary set, $\forall l$, say $I_i \cap I_{i'}$ & $I_j \cap I_{j'} = \phi$ for $i \neq i'$, $j \neq j'$.

If $\exists i \in \{1,2,\dots,k\}$ & $j \in \{1,2,\dots,m\}$ with $I_i = (0,a)$ (or $(0,a]$) and $I_j = (0,b)$ (or $(0,b]$) at the same time, then $A \cap B \neq \phi$, $\divideontimes$, so in the case of $A \cap B = \phi$, there is only A or B contains the interval like $(0,c)$ (or $(0,c]$) for some c, thus

$$\phi(A \cup B) = \phi\left(\bigcup_{i=1}^{k} I_i \cup \bigcup_{j=1}^{m} I_j\right) = \phi\left(\bigcup_{i=1}^{k} I_i\right) + \phi\left(\bigcup_{j=1}^{m} I_j\right) = \phi(A) + \phi(B).$$

- <u>nonregular</u>

Consider $(0, c]$, then given $\frac{c}{2} > 0$, there is no closed set $F \subseteq (0, c]$ s.t. $\phi((0,c]) - \phi(F) \leq \frac{c}{2}$ for $\phi(0,c] = 1+c$ and $\phi(F) \leq 1 \Rightarrow \phi(0,c] - \phi(F) \geq c$.

- <u>ϕ cannot be extended to countably additive set</u>

Consider $(0, \frac{1}{2}] = \bigcup_{n=1}^{\infty} (\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, then

$\phi(0, \frac{1}{2}] = 1 + \frac{1}{2} = \frac{3}{2} \neq \frac{1}{2} = \sum_{n=1}^{\infty} \phi((\frac{1}{2^{n+1}}, \frac{1}{2^n}])$.

7. Given any $a \in \mathbb{R}$, we want to show that

$$E = \{x \mid f(x) > a\}$$

is measurable.

Let $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ s.t. $x_n \searrow a$, then

$E = \bigcup_{k=1}^{\infty} \{x \mid f(x) > x_n\}$, and each $E_k = \{x \mid f(x) > x_k\}$

is measurable, thus E is measurable set.

$\therefore$ f is measurable.

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8. ($\Rightarrow$) Suppose f meas. on X, then $\forall a \in \mathbb{R}$,

$E_a := \{x \mid f(x) > a\}$ is measurable.

Then for $f|_A$ & $f|_B$, $\{x \mid f|_A(x) > a\}$ and

$\{x \mid f_B(x) > a\}$ are $E_a \cap \underset{\mathcal{M}}{\underset{\Uparrow}{A}}$, $E_a \cap \underset{\mathcal{M}}{\underset{\Uparrow}{B}}$, respectively.

Thus $f|_A$ & $f|_B$ meas.

($\Leftarrow$) Suppose $f|_A$ & $f|_B$ meas. on A & B respectively. Then given any $a \in \mathbb{R}$, we want to show $E_a = \{x \mid f(x) > a\}$ is measurable.

Since $X = A \cup B$, then

$$\begin{aligned} E_a &= (A \cap E_a) \cup (B \cap E_a) \\ &= \{x \in A \mid f(x) > a\} \cup \{x \in B \mid f(x) > a\} \\ &= \{x \mid f|_A(x) > a\} \cup \{x \mid f|_B(x) > a\} \in \mathcal{M}. \end{aligned}$$

$\therefore$ f is measurable on X, done.