HW3 Solutions

(1) If not.

$$E = \{x \in X \mid f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n ,$$

where $E_n = \{x \in X \mid f(x) > \frac{1}{n}\}$.

If $\mu(E) > 0 \Rightarrow$ some $\mu(E_n) > 0$

(otherwise $\mu(E) \le \sum_{n=1}^{\infty} \mu(E_n) = 0$)

$$\int_X f \, d\mu \underset{f \geq 0}{\geq} \int_{E_n} f \, d\mu > \frac{1}{n} \mu(E_n) > 0 \quad \times$$

$\therefore \mu(E) = 0$. QED

(2) We first prove ⓑ, which helps to prove other parts:

ⓑ Given $f \le g$, we write

$$X = E_1 \cup E_2 \cup E_3 ,$$

where
$E_1 = \{f, g \geq 0\}$
$E_2 = \{f < 0, g \geq 0\}$
$E_3 = \{f, g < 0\}$.

and $\int_{E_i} f \, d\mu \le \int_{E_i} g \, d\mu \quad \forall i$ (show)

on E_1,

$$\left\{ \int_{E_1} s \;\middle|\; \begin{matrix} 0 \le s \le f \\ \text{simple} \end{matrix} \right\} \subseteq \left\{ \int_{E_1} s \;\middle|\; \begin{matrix} 0 \le s \le g \\ \text{simple} \end{matrix} \right\}$$

$$\therefore \underbrace{\sup(LHS)}_{\int_{E_1} f \, d\mu} \le \underbrace{\sup(RHS)}_{\int_{E_1} g \, d\mu}$$

the same result holds on E_3, with f, g replaced by $-f, -g \geq 0$

cont. on E_2, $\int_{E_2} g \, d\mu \geq 0$. ②

and $f = -(f^-) \geq 0$ $\int_{E_2} f \, d\mu = -\underbrace{\int_{E_2} f^- \, d\mu}_{\geq 0} \leq 0$

By definition

$\therefore \int_{E_2} f \, d\mu \leq 0 \Rightarrow \int_{E_2} f \, d\mu \leq \int_{E_2} g \, d\mu$

ⓑ ✓.

ⓐ then follows straight from ⓑ.

ⓒ. We prove the case $f \geq 0$, and general case follows by writing $cf = \underbrace{cf^+}_{\geq 0} - \underbrace{cf^-}_{\geq 0}$

$f \geq 0$, the case for simple function is trivial.

For $c \geq 0$

$$c \cdot \sup \left\{ \int_X s \, d\mu \;\middle|\; 0 \leq \underset{\text{simple}}{s} \leq \right\} = c \int_X f \, d\mu$$

$$= \sup \left\{ c \int_X s \, d\mu \;\middle|\; 0 \leq \underset{\text{simple}}{s} \leq f \right\}$$

$$= \sup \left\{ \int_X cs \, d\mu \;\middle|\; 0 \leq \underset{\text{simple}}{s} \leq f \right\}$$

$$= \sup \left\{ \int_X s' \, d\mu \;\middle|\; 0 \leq \underset{\text{simple}}{s'} \leq cf \right\}$$

$$= \int_X cf \, d\mu$$

For $c < 0$, consider $\int_X (-c) f = -c \int_X f \, d\mu$

$cf \leq 0$. $\int cf = -\int -cf$ $\quad = -\int_X cf \, d\mu$

✓

③

ⓓ $\int_X f K_E \, d\mu = \underbrace{\int_{X \setminus E} f K_E \, d\mu}_{\substack{\parallel \text{ since } K_E = 0 \\ 0 \quad \text{on } X\setminus E}} + \int_E f K_E \, d\mu$

$\parallel \quad K_E = 1$ on E

$\int_E f \, d\mu$

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ⓔ $f \in \mathcal{L}(X) \quad \left(\int_X |f| < \infty \right)$

$$\int_E |f| \, d\mu = \int_X |f| K_E \quad (\text{by ⓓ})$$

$$\underset{0 \le K_E \le 1}{\le} \int_X |f| < \infty$$

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(3) Rudin Exercise 8.

$f \in \mathcal{R}([a,b])$, $\quad F(x) = \int_a^x f(t)\,dt$

$\hookrightarrow f$ cont. a.e. on $[a,b]$

$\overset{\text{Fund. thm. of calc.}}{\Rightarrow}$ $F(x)$ diff. a.e on X

and $F'(x) = f(x)$

(Rudin Exercise 12) Note: $g(x)$ is defined since $g(x,y)$ cont. in y and integrable

(4) Yes. let $x_n \to x$ and $f_n(y) = f(x_n, y)$

$$\lim_{n\to\infty} \left(g(x_n) - g(x) \right) = \lim_{n\to\infty} \int_{[0,1]} [f(x_n,y) - f(x,y)]\,dy$$

$$= \lim_{n\to\infty} \int_{[0,1]} [f_n(y) - f(y)]\,dy$$

$|f_n(y) - f(y)|$
$\le |f(x_n,y)| + |f(x,y)|$ DCT
$\le 2 \in \mathcal{L}^1([0,1])$

$$\int_{[0,1]} \underset{\substack{\parallel \\ 0 \\ \text{since } g \text{ cont. in } x}}{\lim_{n\to\infty} \cdots} \, dy = 0$$

$\therefore g$ cont. at x

(5) Rudin Exercise 11 ④

Repeat the proof for the completeness of $\mathcal{L}^p$ in class with $p=1$.

(6) To be discussed later.

(7) $\forall \varepsilon>0$, $\exists$ simple function $s = \sum_{i=1}^{N} a_i K_{E_i}$ with $0 \le s \le f$

s.t. $\int_X f\,d\mu - \frac{\varepsilon}{2} \le \int_X s = \sum_{i=1}^{N} a_i \mu(E_i) \quad \cdots ⊛$

Since μ is regular, (by Prob 3 of HW2) for each i, $\exists$ $E_i' \subset E_i$

s.t. $\mu(E_i) < \mu(E_i') + \frac{\varepsilon}{2N}$, $E_i' \cap E_j' = \phi$ $\forall i \ne j$

Let $E = \bigcup_{i=1}^{N} E_i'$

$$\text{RHS } ⊛ < \sum_{i=1}^{N} a_i \mu(E_i') + \frac{\varepsilon}{2}$$

$$= \int_E s\,d\mu + \frac{\varepsilon}{2}$$

Since $0 \le s \le f$ on E clearly

$$\le \int_E f\,d\mu + \frac{\varepsilon}{2} \implies \int_E f\,d\mu > \int_X f\,d\mu - \varepsilon$$

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