Name and Student ID's: ______________________________

Homework 4, Undergraduate Analysis

Refer to class notes for notions used in this homework. $\mathcal{H}$ will be used to denote a Hilbert space and $\|\cdot\|$ is the norm induced by inner product $<\cdot,\cdot>$. The metric on $\mathcal{H}$ is given by this norm.

1. Prove that

$$\|\cdot\| : \mathcal{H} \to K$$

is continuous.

2. Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, define *product norm* on $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$\|(x_1, x_2)\| = \max(\|x_1\|_{\mathcal{H}_1}, \|x_2\|_{\mathcal{H}_2}).$$

(a) Prove that this norm is equivalent to

$$\|(x_1, x_2)\|' = \sqrt{\|x_1\|^2_{\mathcal{H}_1} + \|x_2\|^2_{\mathcal{H}_2}}.$$

(b) Prove that vector space addition

$$+ : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$$

is continuous with respect to product norm.

(c) Prove the scalor multiplication

$$\cdot : K \times \mathcal{H} \to \mathcal{H}$$

is continuous with respect to product norm.

3. Prove the continuity of inner product: If $x_n \to x$ and $y_n \to y$ then $< x_n.y_n > \to < x, y >$.

4. Prove the parallelogram law stated in class.

5. Prove the Pythagorean's Theorem stated in class.

6. Let $E \subset \mathcal{H}$. Prove that $(E^\perp)^\perp$ is the smallest closed subspace containing E. That is, if F is a closed subspace containing E, then $(E^\perp)^\perp \subset F$.

7. Here we study a vector space *not* isomorphic to its dual. Define

$$E = \mathbb{R}^{(\mathbb{N})} := \{\{a_i\}_{i=1}^{\infty} \mid a_i \in \mathbb{R} \textit{ and } \exists N \textit{ such that } a_i = 0 \forall i > N\}$$

and

$$V = \mathbb{R}^{\mathbb{N}} := \{\{a_i\}_{i=1}^{\infty} \mid a_i \in \mathbb{R}\}.$$

Clearly, the set $\{e_i\}_{i=1}^{\infty}$, where e_i is the sequence with $a_i = 1$ and $a_j = 0 \; \forall j \neq i$ is a basis for E.

(a) Prove that E is *not* isomorphic to V.

(b) Prove that V is isomorphic to E^* and conclude that E is not isomorphic to E^*.

8. Prove that for any $A \neq \emptyset$, $l^2(A)$ is complete.