Differential Geometry: Final Exam Problem List

1. The space of n matrices, $Mat(n, \mathbb{R})$, is a vector space of dimension n^2 and therefore a smooth manifold. Let $GL(n, \mathbb{R})$ be the subset of invertible $n \times n$ matrices, and $O(n, \mathbb{R})$ be the set of orthogonal matrices:

$$O(n, \mathbb{R}) := \{A \in Mat(n, \mathbb{R}) \mid AA^T = Id\}.$$

   (a) Prove that $GL(n, \mathbb{R})$ is a smooth manifold. What is its dimension?

   (b) Prove that $O(2, \mathbb{R})$ is a smooth manifold. What is its dimension?

2. Given a closed 1-form ω on a smooth manifold M,

   (a) prove that if M is simply connected, γ_1 and γ_2 be two paths starting and ending at the same points, then

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

   (b) show the conclusion of part (a) is false if M is not simply connected.

3. Let M be a compact manifold, N a connected manifold, and $F : M \to N$ is smooth.

   (a) Prove that F is a closed map.

   (b) Prove that if F is a submersion, then N must also be compact.

4. Prove that $T\mathbb{S}^1$ is trivial.

5. Show that every manifold M admits a smooth embedding into some Euclidean space.

6. Let $U = (0, \pi) \times (0, 2\pi)$ and $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

   Compute $\int_U f^*\omega$ for

$$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy.$$

7. Let

$$\alpha = \left(3x^2 \cos y + e^{xy}\right)\, dx \wedge dy + 17x^3 dx \wedge dz + 3\left(x + yz^7 + \cos z\right)\, dy \wedge dz$$

   be a two form on $\mathbb{R}^3$. Compute

$$\int_{\mathbb{S}^2} \alpha,$$

   where $\mathbb{S}^2$ is given by standard orientation determined by outward normal vector.

8. Let M be a compact, oriented and connected manifold with boundary. Let $\iota : \partial M \to M$ be inclusion, α and β be k form and $n-k-1$ form on M, respectively. Moreover assume that $\iota^*\beta = 0$. Prove that

$$\int_M d\alpha \wedge \beta = (-1)^{k+1} \int_M \alpha \wedge d\beta.$$

9.

10.