Note 0 - Preliminaries on Point Set Topology

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1 Definitions

In this note, we define essential topological definitions needed to define manifolds. Topology is a subject that rigorously constructs the abstract concepts of space, position, and continuity via sets and logics. A *space* X is a set whose elements, if any, is called *point* (e.g. metric spaces). For two points $x \neq y \in X$, we often ask how different, or how *far*, these two points are *separated* from each other. For a metric space (X, d), the notion of *nearness* is described precisely by the metric function and the measure of difference is known as the *distance*. A metric then leads to definitions of open subsets. Once open subsets are known, other *topological* definitions such as compactness, connectedness, continuity, convergence,... etc can be established. Note that none of these topological definitions involves metric, if open sets are known in advance. Therefore, for a general space, we *define* open subsets and the *topological structure* of the space is determined by these open subsets.

The collection $\mathcal{T}$ of open subsets of X is called a *topology* of X and $(X, \mathcal{T})$ is called a *topological space*. Of course, the collection $\mathcal{T}$ must satisfy basic properties of open sets, proved for a metric space, and are now *axiomatized*.

The note is a summary of Appendix A of our textbook.

Definition 1.1 (Topological Space)**.**

It is straightforward to verify, by de-Morgan's laws, that closed sets are closed under arbitrary intersection and finite union. One also notes that a topology is equivalently determined by its closed subsets.

Every set X has two obvious topologies. The first one is called *trivial* topology, where $\mathcal{T} = \{\emptyset, X\}$. On the other extreme, it has *discrete* topology, where $\mathcal{T} = \mathcal{P}(X)$. That is, every subset is open. All metric spaces (X, d) are topological spaces, for which we have already seen many example in standard advanced calculus courses. Here are a few non-intuitive simple examples (or non-examples) here.

Example 1.2.

Example 1.3 (Cofinite Topology)**.**

Example 1.4.

There are many more nontrivial and interesting examples of topological spaces. Very often the topology is defined not only to construct geometric structure, but also algebraic or analytic features into the space.

The first concept of nearness given by a topology is the concept of *separation*. Separation of two objects (points or subsets) is roughly defined as two disjoint open sets, each containing one of the two. There are five common levels of separations for a topology denoted by T_j, $j = 0, \ldots, 5$.

Definition 1.5.

One may quite easily prove that a space is T_1 if and only if singletons are closed. A T_2 space is known as a *Hausdorff* space, a T_3 space is known as a *regular* space, and a T_4 space is known as a *normal* space. It is quite obvious that metric spaces satisfy all these separation axioms. However, they are not true in general. Example 1.4 is an example of non-Hausdorff space.

All definitions we have learned in advanced calculus that involve only open sets (and others derived from them) are carried over here without difficulty. One also observes that continuity and convergence may also be easily defined for topological spaces.

Exercise 1.6. State the definition for

- A map between topological spaces $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ to be continuous.
- Define a sequence $\{x_i\}$ in a topological space $(X, \mathcal{T})$ and what it means for x_i to converge to x.

2 Base and Countability

Given a topological space $(X, \mathcal{T})$, a *base* for $\mathcal{T}$ plays similar role as basis for a vector space.

Definition 2.1. A *base* for $\mathcal{T}$ is a subcollection $\mathcal{B} \subset \mathcal{T}$ so that every nonempty open set $U \in \mathcal{T}$ is a union of elements in $\mathcal{T}$.

For a metric space, it is well known that the collection of all open balls form a base. The definition is equivalent to the following property that is easier to check, but less reflective of the word "base".

Proposition 2.2. *$\mathcal{B} \subset \mathcal{T}$ is a base if and only if*

Exercise 2.3. Check the equivalence.

The condition above gives the local version of base:

Definition 2.4. For $x \in X$, a *neighborhood base* for $\mathcal{T}$ at x is a collection $\mathcal{B}_x$ of open sets containing x so that

$$x \in U \in \mathcal{T} \;\Rightarrow\; \exists V \in \mathcal{B}_x \text{ so that } x \in V \subset U.$$

Obviously, a base for $\mathcal{T}$ consists of neighborhood basis at every point. In studying manifolds, we focus on spaces whose topologies are generated by "small" bases.

Definition 2.5. A topological space $(X, \mathcal{T})$, is called

Exercise 2.6. If $(X, \mathcal{T})$ is second countable, then every open cover has a countable subcover. (It is "σ-compact")

Just as base of topology is analogous to basis of a vector, we have an analogous construction of "span" for topological space. Recall that, given a subset E of a vector space, the *span* of E is the smallest subspace that contains E. Or, equivalently, it is the intersection of all vector subspaces that contain E. For a general set X, we have

Definition 2.7. Let $\mathcal{E} \subset \mathcal{P}(X)$ be a collection of subsets containing $\emptyset$ and X, the topology *generated* by $\mathcal{E}$, or $\mathcal{T}(\mathcal{E})$, is the smallest topology containing $\mathcal{E}$. That is, if $\mathcal{E} \subset \mathcal{T}$ and $\mathcal{T}$ is a topology, then $\mathcal{T}(\mathcal{E}) \subset \mathcal{T}$.

It is not difficult to show that any intersection of topologies is still a topology. Therefore, $\mathcal{T}(\mathcal{E})$ is, abstractly, the intersection of all topologies containing $\mathcal{E}$. More explicitly, we can prove

Proposition 2.8. *Given $\mathcal{E} \subset \mathcal{P}(\mathcal{E})$ with $\emptyset, X \in \mathcal{E}$,*

$$\mathcal{T}(\mathcal{E}) = \{\emptyset, X\} \bigcup \{\cup_{B \in \mathcal{B}} B\}_{\mathcal{B} \subset \mathcal{F}},$$

where F is the set of all finite intersections of elements in $\mathcal{E}$.

Essentially, it says that $\mathcal{T}(\mathcal{E})$ consists of $\mathcal{E}$ and all additional elements needed to be a topology.

3 Induced Topology

Given two sets X, Y with certain relations, there are usually natural ways to define topology on one space from another respecting their relations. Most of them can be described as *pullback topology* for maps.

Given a set X, a topological space $(Y, \mathcal{T})$ and a map $f : X \to Y$, consider the collection of subsets of X

$$f^*\mathcal{T} := \{f^{-1}(U) \mid U \in \mathcal{T}\} \subset \mathcal{P}(X).$$

It is straightforward to check that $f^*\mathcal{T}$ is a topology on X. Moreover, it is evident that $f : (X, f^*\mathcal{T}) \to (Y, \mathcal{T})$ is a continuous map. In another word, we define a topology on X that *makes* the map continuous. This construction gives rise to three standard induced topologies.

Example 3.1 (Subspace Topology).

Example 3.2 (Quotient topology).

Example 3.3 (Product Topology).