LECTURE 2 - Definitions and Examples of Smooth Manifolds

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After learning the hopelessly abstract definitions of topological spaces, we study a certain kind of spaces that we can "do some interesting math" on them. To "do some math", we need to be able to, at least locally, identify the topological space with a field homeomorphically. A simple word for this identification is "coordinate". To do some interesting math, we need to, in addition, tell what is "smooth". In another words, we want to do some calculus on our topological space. For this purpose, the coordinates should be identified with spaces at least as complete as $\mathbb{R}^n$. Moreover, the local identification must agree on overlaps so we have an unambiguous definition for smooth maps.

1 Topological Manifolds

We start with the primitive definitions of topological manifolds. All examples below are Hausdorff and second countable and we leave the readers to check those technical details.

Definition 1.1. A *real topological manifold* of dimension n is a topological space M that is Hausdorff, second countable and *locally Euclidiean.* That is, for each $p \in M$, there is a pair (U, φ) so that

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You may check that it loses no generality to assume that $\tilde{U}$ is an open ball in $\mathbb{R}^n$. The pair (U, φ) is called a *coordinate chart* around p. The coordinate is usually denoted by

$$\varphi(p) = (x^1(p), \ldots, x^n(p)).$$

The coordinate representation for p is certainly not unique. It is possible for p to lie in two coordinate charts (U, φ) and (V, ψ). The intersection $U \cap V$ is mapped homeomorphically to two open sets $\varphi(U \cap V) \subset \tilde{U}$ and $\psi(U \cap V) \subset \tilde{V}$ by

φ and ψ. These two open sets are clearly homeomorphic via the homeomorphism $\psi \circ \varphi^{-1}$ restricting on $\varphi(U \cap V)$.

The restriction of the map $\psi \circ \varphi^{-1}$ is called the *transition map*, and we conclude that every point on M has a well defined coordinate in $\mathbb{R}^n$ up to homeomorphism.

We introduce several basic examples of topological manifolds. Some are visually obvious (even trivial). Some are more abstract.

Example 1.2. *$M = \mathbb{R}^n$ with a single coordinate chart $(U, \varphi) = (\mathbb{R}^n, Id)$ is trivially a topological manifold.*

Example 1.3 (Graph of Continuous Functions)**.** *Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^k$ be a continuous map. The* graph *of F*

$$Gr(F) := \{(x, F(x))\} \subset \mathbb{R}^n \times \mathbb{R}^k$$

with subspace topology is a topological manifold of dimension n. It is again covered by one coordinate chart $U = Gr(F)$ with homeomorphism $\varphi : Gr(F) \to U$ given by $\varphi(x, F(x)) = x$. This map is clearly bijective and continuous since it is the restriction of projection map $\pi_1(x, y) = x$ to $Gr(F)$. Its inverse, $\varphi^{-1}(x) = (x, F(x))$ is also continuous since all components are.

Example 1.4 (Spheres)**.** *The unit n-sphere*

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

with subspace topology is a topological manifold. It is covered by $2n+2$ coordinate charts

$$U_i^+ := \{(x^1, \ldots, x^{n+1}) \mid x^i > 0\}$$

and

$$U_i^- := \{(x^1, \ldots, x^{n+1}) \mid x^i < 0\}$$

for $i = 1, \ldots, n+1$. It is clear that each U_i^+ is the graph of continuous map

$$x^i = \sqrt{1 - \sum_{j=1, j\neq i}^{n+1} (x^j)^2}$$

and U_i^- *is the graph of continuous map*

$$x^i = -\sqrt{1 - \sum_{j=1, j\neq i}^{n+1} (x^j)^2}.$$

The coordinate map is then simply

$$\varphi_i^{\pm}(x^1, \ldots, x^{n+1}) = (x^1, \ldots, \hat{x^i}, \ldots, x^{n+1}).$$

The next example is more abstract, but extremely important.

Example 1.5 (Real Projective Space)**.** *The* real projective space *is the space of 1-dimensional subspace, or lines through origin, in* $\mathbb{R}^{n+1}$*. Precisely, consider* $\mathbb{R}^{n+1}\backslash\{0\}$ *and define equivalence relation* $x \sim y \iff x = \lambda y$ *for some* $\lambda \neq 0$*. (Check the equivalence). We define the real projective space to be*

$$\mathbb{RP}^n := \mathbb{R}^{n+1}\backslash\{0\}/\sim$$

with quotient topology. We denote the equivalence class with representative $(x^1, \ldots, x^{n+1})$ *by* $[x^1 : \ldots : x^{n+1}]$*. Clearly,* $[x^1 : \ldots : x^{n+1}] = [\lambda x^1 : \ldots : \lambda x^{n+1}]$ *for all* $\lambda \neq 0$*, and the representation is known as* homogeneous coordinate*.*

Let $\pi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{RP}^n$ *be the quotient map,* $\tilde{U}_i$ *be the subset of* $\mathbb{R}^n$ *with* $x^i \neq 0$*, and* $U_i = \pi(\tilde{U}_i)$*. These subsets are open in* $\mathbb{RP}^{n+1}$ *(check it). Each* U_i *is a local coordinate chart with coordinate map*

$$\varphi_i([x^1 : \ldots : x^{n+1}]) = \left(\frac{x^1}{x^i}, \ldots, \frac{\widehat{x^i}}{x^i}, \ldots, \frac{x^{n+1}}{x^i}\right).$$

Readers may check that this map is well-defined, continuous (since $\varphi_i \circ \pi$ *is), with continuous inverse*

$$\varphi_i^{-1}(u^1, \ldots, u^n) = [u^1 : \ldots : u^{i-1} : 1 : u^i : \ldots : u^n].$$

These coordinate charts (U_i, φ_i)*, for* $i = 1 \ldots, n+1$*, form an open cover of* $\mathbb{RP}^n$ *and equip it with a structure of topological manifold.*

Let's try to visualize this space for $n = 3$ and justify its name.

2 Smooth Manifolds

We now raise our standard and discuss manifolds that look "smooth and curvy". Such spaces allow us to do usual things we do in calculus at least locally, such as tangent lines, tangent planes, and computing curvatures, ...etc. Let's first recall (or agree on) some definitions from advanced calculus.

Definition 2.1. Given an open set $U \subset \mathbb{R}^n$, a map $F : U \to \mathbb{R}^m$ is smooth, or C^∞, if partial derivatives exist to any order and are continuous. For $V \subset \mathbb{R}^m$, $F : U \to V$ is called a *diffeomorphism* if is bijective with smooth inverse.

A diffeomorphism is clearly a homeomorphism. Moreover, it preserves smooth functions on U or V: any smooth function $f : V \to \mathbb{R}$ (or $U \to \mathbb{R}$) gives rise to a smooth function on U (or on V) by composing with F (or F^{-1}). In short, a diffeomorphism preserves the smooth structure of two spaces, and this is the additional requirement we are imposing on *smooth manifold.*

Let's define smooth structure on topological manifolds. A smooth structure is a rule to determine how derivatives are taken and what functions are smooth. Since every $p \in M$ has a neighborhood U that can be identified with an open set $\tilde{U} \subset \mathbb{R}^n$ via homeomorphism φ, it is natural to define derivative of a function f at p to be derivative of $f_U := f \circ \varphi^{-1} : \tilde{U} \to \mathbb{R}$ (called the *coordinate representation* of f on U), and say f smooth at p if f_U is. The ambiguity, however, arises on *overlaps* of coordinate charts. If (U, φ) and (V, ψ) are both coordinate charts near p, there will be *two* identifications with open sets in $\mathbb{R}^n$ and therefore two rules of differentiation: f_U and f_V. They are related by transition maps $\psi \circ \varphi^{-1}$, which does not necessarily preserve smoothness:

Example 2.2. *$\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\varphi(u, v) = (u^{\frac{1}{3}}, v^{\frac{1}{3}})$ is a homeomorphism. $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x$ is clearly smooth, but $f \circ \varphi(u, v) = u^{\frac{1}{3}}$ is not smooth at $(0, 0)$.*

Defining smoothness as above near p evidently depends on the choice of coordinate, which does not fit well with our intuition. The dependence is eliminated if we require in addition the transition maps between any two coordinate charts to be *diffeomorphism.* Two coordinate charts are called *smoothly compatible* if the transition map between them is diffomorphism in $\mathbb{R}^n$. A *smooth atlas* is a collection of smoothly compatible charts that cover M, and

Definition 2.3 (Smooth Manifold)**.** A *smooth manifold M* is a topological manifold with a smooth atlas.

With a smooth atlas, we can now define smooth functions on M.

Definition 2.4 (Smooth Function)**.** Given a smooth manifold M and $f : M \to \mathbb{R}$, f is smooth at p if for some coordinate chart U of p, $f_U : \tilde{U} \to \mathbb{R}$ is smooth in the usual sense. f is smooth on M, or $f \in C^\infty(M)$ if it is smooth at every $p \in M$.

Of course a manifold may very well have more than one smooth atlas. Two atlas are *equivalent* if they define the same smooth functions. That is, a function is smooth in one atlas if and only if it is smooth in another. We may combine all equivalent atlas and define a *maximal* atlas for that structure. A *smooth structure* is then a maximal smooth atlas. Some topological manifolds have unique smooth structure (up to equivalence), some might have many, and some might have none. They are all very interesting problems to study, but we will not go into further details.

Let's check that Example 1.2 - 1.5 are all smooth manifolds. The first two examples require no work since the chart is global and there is no transition map. For the sphere in Example 1.4, one can readily compute that the transition maps on $U_i^\pm \cap U_j^\pm$. For $i < j$, we have

$$\varphi_i^\pm \circ (\varphi_j^\pm)^{-1}(u^1, \ldots, u^n) = (u^1, \ldots, \widehat{u^i}, \ldots, \pm\sqrt{1 - \|u\|^2}, \ldots, u^n)$$

and similar expressions hold for $i > j$ (just switch the hat and square root). For $i = j$, the transition maps are identities. On $U_i^\pm \cap U_j^\pm$, it is clear that $\|u\| < 1$ and the map above is a diffemorphisms on their domains and ranges.

For the real projective space in Example 1.5, the transition maps on $U_i \cap U_j$ are

$$\varphi_i \circ \varphi_j^{-1}(u^1, \ldots, u^n) = \left(\frac{u^1}{u^j}, \ldots, \frac{u^{j-1}}{u_j}, \frac{u^{j+1}}{u_j}, \ldots, \frac{u^{i-1}}{u_j}, 1, \frac{u^i}{u_j}, \ldots, \frac{u^n}{u_j}\right)$$

which are diffeomorphisms on their domains and ranges since the denominator is never 0.

We give a few more examples of smooth manifolds.

Example 2.5 (Finite Dimensional Vector Spaces)**.** *Given a finite dimensional real vector space V, we may define a norm and therefore a topology on it. Recall that being finite dimensional, all norms are equivalent and therefore the topology is independent of the choice of norm. Let $\{v_i\}_{i=1}^n$ be a basis of V. The space V is homeomorphic with $\mathbb{R}^n$ via homeomorphism $\varphi(x) = \varphi\left(\sum_{i=1}^n x^i v_i\right) := (x^1, \ldots, x^n)$. The single chart (E, φ) then defines a single coordinate chart and a smooth structure. One can readily check that this smooth structure is independent of choice of basis since a different basis amounts to a change of basis matrix, which is a linear and invertible map on $\mathbb{R}^n$ and therefore diffeomorphic.*

Example 2.6 (Matrices)**.** *By previous example, the space $Mat(m \times n, \mathbb{R})$ of all real $m \times n$ matrices is a smooth manifold since it is a real vector space of dimension nm.*

Example 2.7 (Open Submanifolds)**.** *Given a smooth manifold M and an open subset $U \subset M$, U then has a smooth atlas given by*

$$\{(W \cap U, \varphi_W|_U) \mid (W, \varphi_W) \text{ smooth chart for } M\}.$$

Therefore any open subset of a smooth manifold is also a smooth manifold.

Example 2.8 (General Linear Group)**.** *By the previous two examples, the open subset*

$$GL(n,\mathbb{R}) = \{A \in Mat(m \times n, \mathbb{R}) \mid A \text{ invertible } \}$$

is a smooth manifold. Moreover, it is a group under matrix multiplication. Multiplication and inverse operations are also smooth with respect to its smooth structure. This is an example of Lie group.

Example 2.9 (Implicitly Defined Submanifolds)**.** *Consider a smooth map* $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$*, written as* $F(x, y)$ *for* $x \in \mathbb{R}^n, y \in \mathbb{R}^m$*, with* $F^{-1}(0) \neq \emptyset$*. The derivative of* F *is written as*

$$DF_{(u,v)} = (DF_x \mid DF_y)\,|_{(u,v)}$$

where DF_x *consists of partial derivatives with respect to first* n *variables and* DF_y *with respect to the remaining* m*.* DF_y *is a linear operator on* $\mathbb{R}^m$*. If* DF_y *is invertible on* $F^{-1}(0)$*, then* $F^{-1}(0)$ *is an* n*-dimensional submanifold of* $\mathbb{R}^{n+m}$*.*

Indeed, by the Implicit Function Theorem, $F^{-1}(0)$ *is covered by open subsets* $\{U_\alpha\}$ *and for every* α*, there exists a unique smooth map* $g_\alpha : U_\alpha \to \mathbb{R}^m$ *so that* $U_\alpha = \{(x, g_\alpha(x))\}$*. Therefore,* U_α *is a coordinate chart with coordinate map* $\varphi_\alpha(x, y) = x$*. It is then clear that transition maps on two overlapping coordinate charts are identities, and therefore* $F^{-1}(0)$ *is a smooth manifold.*

3 Manifolds with Boundary

At this point we have been working with manifolds without boundary, on which every point is an interior point. The *model space* at each point is an open ball in $\mathbb{R}^n$, or the entire $\mathbb{R}^n$. These spaces are at least insufficient for the formulation of the fundamental theorem of calculus, which relates the integral over a space to another integral over the boundary.

The model space for a manifold with boundary is the closed upper half plane

$$\mathbb{H}^n := \{(x^1, \ldots, x^n) \mid x^n \geq 0\}.$$

An *n-dimensional topological manifold with boundary* is a second countable, Hausdorff topological space in which every point $p \in M$ has a neighborhood (cahrt) that is either homeomorphic to an open subset of $\mathbb{R}^n$, or a relative open subset in $\mathbb{H}^n$ (that is, $U \cap \mathbb{H}^n$, where U open in $\mathbb{R}^n$. Points of first type are called an *interior point*, or $p \in \ Int\ M$, and the corresponding chart (U, φ) is called an interior chart. Those of second type are called *boundary points*, or $p \in \partial M$ with boundary chart (U, φ). It is intuitively clear, although not straightforward to prove that a point is *exactly* one of the two types:

Theorem 3.1. *M is the disjoint union of $IntM$ and ∂M.*

We remind the readers that ∂M is in general not the same as the actual topological boundary of M. See textbook for clarifications.

Next we construct smooth structure on manifolds with boundary. First, we recall the definition of smooth maps on arbitrary subsets $A \subset \mathbb{R}^n$:

Definition 3.2. Given $A \subset \mathbb{R}^n$, a map $F : A \to \mathbb{R}^k$ is smooth if there is an open subset $U \subset \mathbb{R}^n$ that contains A, and a smooth map $\tilde{F} : U \to \mathbb{R}^k$ so that $\tilde{F}|_A = F$.

It basically says that smooth maps on an arbitrary subset are those extendable to a smooth maps defined on a larger open subset. The definition defines smooth functions, and therefore diffeomorphisms, on $\mathbb{H}^n$. A smooth structure for a manifold M with boundary is then identical to smooth structure for M without boundary, except that diffeomorphic transition maps are between subsets in $\mathbb{H}^n$. This defines a *smooth manifold with boundary.*

We conclude this section with a special case of Theorem 3.1:

Theorem 3.3. *Suppose M is a smooth manifold with boundary and $p \in M$. If there is a smooth chart (U, φ) containing p such that $\varphi(U) \subset \mathbb{H}^n$ and $\varphi(p) \in \mathbb{H}^n$, then the same is true for any other smooth chart containing p.*