Smooth Maps

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Now that we have established the concept of local coordinates of a manifold M, we can define smooth maps *between* manifolds that preserve properties we have learned from advanced calculus.

1 Smooth Maps

Just like smooth functions, the smoothness of a map $F : M \to N$ at a point $p \in M$ is defined to be the smoothness of the *coordinate representation* of F at p and $F(p)$:

Definition 1.1. Given smooth manifolds N,M of dimensions n, m respectively, a map $F : N \to M$ is said to be smooth at p, if there are charts (U, φ), (V, ψ) at p and $F(p)$ so that $F(U) \subset V$ and

$$\psi \circ F \circ \varphi^{-1} : \varphi(U)(\subset \mathbb{R}^n) \to \psi(V)(\subset \mathbb{R}^m)$$

is a smooth map in the ordinary sense.

It is now clear that compatibilities of charts on N and M make the definition above independent of choice of coordinate:

The following definitions can now be defined on manifolds:

Definition 1.2. A map $F : N \to M$ is smooth if it is smooth at every point.

Definition 1.3. A map $F : NtoM$ is called a diffeomorphism if it is smooth, bijective, and F^{-1} is smooth.

Constant maps from M to N, identity maps from M to itself, and inclusion maps of an open subset U to M, should all be smooth. Indeed, their coordinate representations are constant maps in Euclidean spaces. Some basic properties should hold true for smooth maps:

Theorem 1.4. *Smooth maps are continuous.*

Theorem 1.5. *Composition of smooth maps is smooth.*

It is also important to point out that smoothness is a local property.

Proposition 1.6. *Let N and M be smooth manifolds and $F : N \to M$ be a map.*

- *If every $p \in M$ has an open neighborhood U such that $F|_U$ is smooth, then F is smooth.*
- *If F is smooth, then its restriction to every open subset is smooth.*

The proposition immediately implies

Corollary 1.7. *Let M and N be smooth manifolds, and $\{U_\alpha\}_\alpha$ be an open cover of M. Suppose that for each α, there exists smooth functions $F_\alpha : U_\alpha \to N$ so that for all α, β,*

$$F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}.$$

Then, there exists a smooth function $F : M \to N$ so that $F_{U_\alpha} = F_\alpha$.

Let's study some example of smooth maps and diffeomorphisms.

Diffeomorphisms are basically the "identifiers" of smooth manifolds without concerns of metrics. Though a manifold may have many distinct smooth structures, they may be related to each other via diffeomorphisms. It is a fascinating (and very deep) subject to classify smooth type of a topological manifold, i.e. smooth structures up to diffeomorphisms.

It is not difficult to show the un-surprising fact that dimensions and boundaries are invariants under diffeomorphisms:

Theorem 1.8. *An m-dimensional smooth manifolds can not be diffeomorphic to an n-dimensional manifolds unless $n = m$.*

Theorem 1.9. *Suppose M and N are smooth manifolds with a diffeomorphism $F : M \to N$. Then $F(\partial M) = \partial N$ and F restricts to a diffeomorphism from $IntM$ to $IntN$.*