Note 3 - Tangent Spaces and Differentials

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Having defined smooth manifolds, smooth functions on them and smooth maps between them, we would like to define derivatives. Throughout the note, N and M are smooth manifolds with dimensions n and m, respectively.

1 Tangent Spaces - Abstract Approach

In calculus, we have learned the concepts of tangent spaces at a point p of a space N, with N being various dimensions (line, surface, ...etc.) In all these cases, tangent space to a space N at point p is the *best approximation* to N by vector space of the same dimension, whose 0 is identified with p. We will therefore define tangent space to a manifold N at p to be some n-dimensional vector space that best approximates. For trivial example, let $N = V$ be an n-dimensional vector space. The tangent space to V at p is simply the translation of V by p. Precisely, it is

$$T_pV = \{(p, v) \mid v \in V\},$$

with vector space operations defined by $a(p, v) = (p, av)$ and $(p, v_1) + (p, v_2) = (p, v_1 + v_2)$. The multiplication and addition on the right are those for V. We have worked on these spaces for $V = \mathbb{R}^n$ when computing DF in advanced calculus, with $T_pV = \mathbb{R}^n_p$, where

$$\mathbb{R}^n_p := \{(p, v) \mid v \in \mathbb{R}^n\}$$

is the vector space with linear structures defined above. We abbreviate the vector (p, v) by v_p. They are referred to as the *geometric tangent vectors at p*.

Nonlinearly, in elementary single variable calculus, we defined tangent line l_p to the graph of a function $F : \mathbb{R} \to \mathbb{R}$ to be the line going through $(p, F(p))$ with slope $F'(p)$. In another words, it is the 1-dimensional vector subspace of $\mathbb{R}^2_{(p,F(p))}$ spanned by $v_p = (1, F'(p))$ whose origin is identified with $(p, F(p)) \in \mathbb{R}^2$. That is naturally the tangent space for the curve $y = F(x)$. Similar situations arise when we have $F : \mathbb{R}^{n-1} \to \mathbb{R}$, where the tangent space to the graph is the $n - 1$ dimensional vector subspace of $\mathbb{R}^n_{(p,F(p))}$ spanned by $\{e_i + \frac{\partial F}{\partial x^i} e_n\}_{i=1}^{n-1}$. These constructions work even for some "smooth" objects in Euclidean spaces that are not graphs of functions:

Nevertheless, one notices that the vector spaces above are all defined on manifolds with a *global* chart: $\mathbb{R}^n$, vector space V, graph of a functions. All the "tangent vectors" are globally identified with ones on $\mathbb{R}^n$ by translating to origin and are all explicitly written accordingly. These descriptions become coordinate dependent when identifications with $\mathbb{R}^n$ are only defined locally. We wonder naturally whether different coordinate really give "the same thing". Or, more fundamentally, what does the phrase "vectors tangent to the space" really mean? As such, we seek alternative definition of tangent vectors that are reasonably coordinate independent and preserve the characteristic property of tangent vectors in $\mathbb{R}^n$, which are then generalized to abstract manifolds.

Take the graph of dimension 1 above for exposition. Every point $P = (p, F(p))$ is globally identified with $p \in \mathbb{R}$. The tangent line is spanned by vector $v_p = (1, F'(p))$. Take a smooth function f on the curve with coordinate representation $\tilde{f}$. Explicitly, we have

$$f(x, F(x)) = \tilde{f}(x).$$

Now take a curve on the graph, which must be of the form

$$\gamma_{v_p}(t) = (\gamma_1(t), F(\gamma_1(t))),$$

with the property that $\gamma_{v_p}(0) = P$ and $\gamma'_{v_p}(0) = v_p$ or equivalently $\gamma(0) = p$ and $\gamma'(0) = 1$. Basic calculus shows that

$$\frac{d}{dt}|_{t=0} f(\gamma_{v_p}(t)) = \tilde{f}'(p).$$

Denote the quantity above by $v_p(f)$. It is not hard to verify that for all smooth functions f, g and $c \in \mathbb{R}$,

- $v_p(f + cg) = v_p(f) + cv_p(g)$.
- $v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$.

Therefore, v_p can be viewed as a linear map on the space of smooth functions satisfying some product rule (a.k.a. the Leibiniz rule). We call such a map a *derivation* at p. One may check that

- The definition $v_p : C^\infty \to \mathbb{R}$ does not involve coordinates.
- Constant functions have zero derivation.

More generally, one can show that for every $a \in \mathbb{R}$,

$$\frac{d}{dt}|_{t=0} f(\gamma_{av_p}(t)) = a\tilde{f}'(p),$$

where $\gamma_{av_p}(t) = (\gamma_a(t), F(\gamma_a(t)))$ with $\gamma_a'(0) = a$. With this, the space of derivations is itself a vector space.

This view of tangent vectors, as actions on smooth functions, continue to make sense for manifolds. We then define, abstractly and coordinate invariantly,

Definition 1.1. Given a smooth manifold N, its *tangent space* at p denoted by T_pN, is the vector space of derivations at p.

The map sending p to $X_p \in T_pN$ is called a derivation on N, which in elementary cases is called vector field.

2 Tangent Spaces - Concrete Constructions

Let's take $N = \mathbb{R}^n$ and show the expected result that $T_pN \simeq \mathbb{R}^n_p \simeq \mathbb{R}^n$ for $p \in \mathbb{R}^n$.

Proposition 2.1. $T_p\mathbb{R}^n \simeq \mathbb{R}^n_p \simeq \mathbb{R}^n$.

We will denote D_i by $\frac{\partial}{\partial x^i}|_p$, or ∂_p^i. They are natural basis for the tangent space of $\mathbb{R}^n$ at p.

One notes that the description above is entirely local. We only need to know the objects involved on some neighborhood of p. Therefore, it entirely makes sense to define a tangent space to a manifold at a point, where local coordinate is given by coordinate chart. Next, we construct basis to tangent spaces of a manifold that are represented by local coordinates.

3 The Differentials

Now that we know what tangent spaces are, we are ready to discuss what "DF" is for a smooth math $F : N \to M$. Recall that in advanced calculus, the derivative of map at a p is defined to be a linear map between tangent spaces at p and $F(p)$ that best approximates the map near them. We will follow this principle to the general case.

Let's start with an abstract definition that seems unrelated. Given a smooth map $F : N \to M$ and $p \in N$, we define the *differential* of F at p to be the linear map $dF : T_pN \to T_{F(p)}M$ by

$$dF(X)(f) = X(f \circ F)$$

for all $f \in C^\infty(M)$. (Verify that $dF(X)$ is indeed a derivation at $F(p)$.) Several basic properties are easy to verify and coincide with usual derivatives. For examples,

- $d(G \circ F) = dG \circ dF$.
- $d(Id_M) = Id_{T_pM}$.
- If F is a diffeomorphism, then dF is a vector space isomorphism.

Differentials are indeed generalizations of ordinary derivatives. Let's dig into the details.

4 Computations in Coordinates

Let's combine all these abstract constructions and express them in terms we are familiar with. Let (U, ϕ) be a coordinate chart of $p \in N$ so that

$$\varphi : U \to \varphi(U) := \tilde{U}$$

is a diffeomorphism (with coordinate representation the identity). Previous results combine together to give isomorphisms between vector spaces below:

$$T_pN \simeq T_pU \simeq T_p\tilde{U} \simeq T_p\mathbb{R}^n \simeq \mathbb{R}^n_p \simeq \mathbb{R}^n,$$

where the second isomorphism is $d\varphi$. We denote

$$\frac{\partial}{\partial x^i}|_p := (d\varphi)^{-1}\frac{\partial}{\partial x^i}|_{\varphi(p)}.$$

More precisely, for smooth function $f \in C^\infty(N)$,

Clearly, $\{\frac{\partial}{\partial x^i}|_p\}_{i=1}^n$ forms a basis for T_pN.

It is then a routine exercise to write out the matrix representation of the linear map $dF_p : T_pN \to T_{F(p)}M$ for a smooth map $F : N \to M$:

Just like in linear algebra, different choices of basis amounts to basis changing isomorphisms:

5 The Tangent Bundle

Let's us finally study something more than "locally advanced calculus/linear algebra" by collecting all the tangent spaces on a manifold M and form a new space. This is an example of "vector bundle", where we attach a vector space to each point of a manifold, all of the same degree, say k. Very often they form new manifolds, and are standard space to describe *configurations spaces* in physics (especially in electromagnetism).

As a set, the tangent bundle is simply

$$TM = \bigsqcup_{p \in M} T_p M.$$

Of course, as sets, TM is always bijective to $M \times \mathbb{R}^m$. In fact, it has a natural *smooth structure* making the projection map smooth.

Theorem 5.1. *For any m-dimensional manifold M, its tangent bundle TM is a smooth manifold of dimension $2m$ so that the projection map π is smooth.*

It is important to know whether TM is anything "new" or just $M \times \mathbb{R}^m$ as manifolds. The latter case is called a *trivial bundle*. For M with a global coordinate chart, we observe that TM is always trivial as $\tilde{\varphi}$ in the proof above gives the required diffeomorphism. This is not true for general manifolds M, but are always true *locally*. That is, $\pi^{-1}(U) \subset TM$ is always diffeomorphic to $U \times \mathbb{R}^m$ via the local coordinate chart. Therefore, just like manifolds, which consist of gluing together Euclidean spaces, tangent bundles (in fact vector bundles in general) consist of gluing together trivial bundles.