Note 4 - Submersions, Immersions, and Embeddings

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1 Introduction

In advanced calculus, we have studied inverse function theorem (or equivalently implicit function theorem). They tell us, for one thing, that a function at a point behaves like its derivative. The derivative, as a linear map between vector spaces, is essentially determined by its rank (dimension of the image) up to choice of basis. From this point of view, the rank of the derivative determines the map up to diffeomorphism.

Since manifolds are locally Euclidean, it is natural to expect that some forms of those results continue to hold true. We will study maps whose differentials have constant rank and explore more properties in the next note.

2 Maps of Constant Rank

The maps of particular interests are ones whose differentials have constant rank throughout the manifold.

Definition 2.1 (Maps of Constant Rank)**.** A smooth map $F : M \to N$ is of *constant rank* r if the linear map $dF_p : T_pM \to T_{F(p)}N$ is of rank r at every point $p \in M$.

The rank of a map is clearly no greater than the minimum of $dimM$ and $dimN$. F has *full rank* if dF_p is equal to $min\{dimM, dimN\}$ for every p.

Definition 2.2 ($dimM \leq dimN$)**.**

Definition 2.3 ($dim M \geq dim N$)**.**

Let's observe some examples:

It is not hard to see that being full rank is an open condition:

Proposition 2.4. *Suppose $F : M \to N$ smooth and dF_p is full ranked, then there exists open subset U around p so that dF is full ranked on U.*

A classical (and very important) theorem in advanced calculus about local behavior of a smooth function versus its derivative is the inverse function theorem. It says that if a map has nonsingular (i.e.e invertible) derivative at a point, then near that point the function is invertible with smooth inverse. In other words, it is a *local diffeomorphism.*

Definition 2.5. A map $F : M \to N$ is a local diffeomorphism if there exists a neighborhood U of $p \in M$ so that $F|_U : U \to F(U)$ is a diffeomorphism, for every $p \in M$.

We have

Theorem 2.6 (Inverse Function Theorem)**.** *Given a smooth map $F : M \to N$ and $p \in M$ so that dF_p is invertible. Then there exists neighborhood U_0, V_0 of p and $F(p)$, respectively, such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Another equivalent form of inverse function theorem is the *rank theorem*:

Theorem 2.7. *Given smooth manifolds M, N with dimensions m and n, and a smooth map $F : M \to N$ with constant rank r, for each $p \in M$, we may choose coordinate charts around p and $F(p)$ so that F has coordinate representation*

$$\hat{F}(x^1, \ldots, x^r, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

We can prove

Theorem 2.8. *Inverse function theorem is equivalent to the rank theorem.*

Rank theorem implies the following global consequence:

Theorem 2.9. *Suppose $F : M \to N$ is a smooth map of constant rank, then*

- *If F is surjective, then it is a submersion.*
- *If F is injective, then it is an immersion.*
- *If F is bijective, then it is a diffeomorphism.*

Injective smooth immersion is not exactly a smooth embedding, but a local one.

Proposition 2.10. *Given an injective smooth immersion $F : M \to N$, it is a smooth embedding if any of the followings is true:*

a *F is open or closed.*

b *M is compact.*

c *F is proper (i.e. $F^{-1}(K)$ is compact for $K \subset N$ compact.)*

Theorem 2.11. *Given smooth map $F : M \to N$, then it is an smooth immersion if and only if every point of M has a neighborhood on which F is a smooth embedding.*

3 Submersion

Submersions between smooth manifolds resemble quotient maps between topological spaces. Recall

Definition 3.1. A map between topological space $\pi : X \to Y$ is called a quotient map if it is surjective, continuous, and Y has quotient topology determined by π.

An important object associated with submersion is the notion of *section*:

Definition 3.2. Given continuous map $\pi : M \to N$, a section of π is a map $\sigma : N \to M$ so that $\pi \circ \sigma = id_N$. If σ is only defined locally, it is called a *local section*

The rank theorem implies the following theorem to characterize submersions:

Theorem 3.3. *Given smooth map $F : M \to N$, then F is a smooth submersion if and only if every point of M is in the image of a smooth local section of F.*

The previous theorem immediately gives rise to the following example of submersion:

Example 3.4. *The projection map* $\pi : TM \to M$ *is a submersion.*

With the theorem, we see that submersion is almost a quotient map:

Theorem 3.5. *Given submersion* $\pi : M \to N$*, then* π *is an open map. It is a quotient map if it is surjective.*

Similar to quotient maps, smooth maps o N are in correspondence with their *lifts* to M:

Theorem 3.6 (Characteristic Property of Surjective Smooth Submersons)**.** *Let* $\pi : M \to N$ *be smooth surjective submersion, and* P *another manifold. Then a map* $F : N \to P$ *is smooth if and only if* $F \circ \pi : M \to P$ *is smooth.*

On the other hand, a smooth map $F : M \to P$ can always be "pushed down" to a map on its quotient, as long as it respects each *fiber* $\pi^{-1}(p)$:

Theorem 3.7. *Let $\pi : M \to N$ be a smooth surjective submersion, and $F : M \to P$ is a smooth map, then there exists a unique map $\tilde{F} : N \to P$ such that $\tilde{F} \circ \pi = F$.*