Note 5 - Submanifolds and Related Topics

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1 Introduction

We continue from Note 4 and study manifolds inside manifolds. Like in linear algebra, where subspaces are images of linear maps. Submanifolds, depending on how well they live inside the ambient spaces, are images of immersions or embeddings. One type of submanifolds of particular interest is the ones with dimension 1 less. They are often related to the level set of smooth functions. We will study their exact occurrence. Not surprisingly, rank theorem (equivalently inverse function theorem) will play central role here. We will also study how tangent space of submanifolds are related to the tangent space of the ambient space. Very important results in this regard are the *Sard's Theorem* and *Whitney's Embedding Theorem*. The first theorem says that there are not too many "bad points" for a smooth map, and the second one says that we can always embed a manifold in an Euclidean space. We will at least address both of them.

2 Embedded Submanifolds

An *embedded submanifold* of a smooth manifold M is a subset $S \subset M$ that is a manifold with subspace topology, and a smooth structure so that the inclusion $\iota : S \hookrightarrow M$ is smooth. M is called the ambient space of S and we define *codimension* of S to be $dimM - dimS$. S is an embedded *hypersurface* if the codimension if 1.

Example 2.1 (Codimension 0 Submanifold)**.**

Example 2.2 (Images of Embeddings)**.**

Since every embedded submanifold is the image of its own inclusion, which by definition is a smooth embedding, we conclude that embedded submanifolds are *precisely* images of smooth embeddings.

Example 2.3 (Slices of Product Manifolds)**.**

Example 2.4 (Graphs)**.**

In fact, every embedded submanifold locally looks like slices.

Definition 2.5. Given a subset S in a smooth manifold M and $k \in \mathbb{N}$, it satisfies the *k-slice condition* if for every $x \in S$, there exists a chart (U, φ) so that

$$\varphi(S \cap U) = \{(x^1, \ldots, x^k, , x^{k+1}, \ldots, x^m) \mid x^{k+1} = c^{k+1}, \ldots, x^m = c^m\}.$$

Theorem 2.6. *Given a k-dimensional embedded submanifold $S \subset M$, then S satisfies local k-slice condition. Conversely, if S satisfies the local k-slice condition, then there is a topological structure of M so that $S \subset M$ is an embedded submanifold.*

The proof is straightforward (although a bit long and tedious) with an application of the rank theorem. We leave it for an exercise.

3 Level Set

A particularly important class of codimension 1 submanifold are those defined by a smooth function. More precisely, they are the subset of the form $f^{-1}(c)$ for some smooth function $f : M \to \mathbb{R}$ and $c \in \mathbb{R}$. For appropriate c, we expect it to be something of dimension 1 less than the ambient space. As we have seen in calculus, the geometric properties (tangent vector, smoothness, ...etc) of $f^{-1}(c)$ is usually determined by the analytic properties of f.

Definition 3.1. Given a smooth function $\Phi : M \to N$, and a point $c \in N$, the *level set* of Φ at c is defined by

$$\Phi^{-1}(c) := \{x \in M \mid \Phi(x) = c\}.$$

Some level sets are nice, and some are not:

Example 3.2.

The same function Φ can have nice and bad level sets for different values of c:

Example 3.3.

The singularities we have observed are largely a consequence of changes in the rank. In fact, we have

Theorem 3.4. *Given a smooth map $\Phi : M \to N$ of constant rank r, then every level set of Φ is an embedded submanifold of codimension r. Moreover, the embedding is proper.*

This theorem resembles the implicit function theorem in advanced calculus. The proof, not surprisingly, is an easy consequence of the rank theorem.

Corollary 3.5. *Every level set of a submersion $\Phi : M \to N$ is a properly embedded submanifold.*

Proof. Exercise. □

One notices that the surjectivity of differential is the only condition needed to prove the previous corollary (for which a submersion satisfies at every point). We define

Definition 3.6. A point $p \in M$ is a *regular point* of Φ if $d\Phi_p : T_pM \to T_{\Phi(p)}N$ is surjective. Otherwise, it is called a *critical point.* A value $c \in N$ is called a regular value if every point of $\Phi^{-1}(c)$ is a regular point. Otherwise, it is called a *critical value.*

The previous corollary is improved to

Corollary 3.7. *For every regular value $c \in N$, the level set $\Phi^{-1}(c)$ is a properly embedded submanifold of dimension $dim M - dim N$.*

Example 3.8 (Sphere)**.**

The map Φ discussed above is called the defining map of the submanifold $S = \Phi^{-1}(c)$. Not every embedded submanifold is of this form. (Existence of global defining function is equivalent to the triviality of certain vector bundle along S). However, every embedded submanifold is *locally* of this form:

Proposition 3.9. *Given a subset S of a smooth manifold M, then S is an embedded submanifold of codimension k if for every point $p \in S$, there is a chart (U, φ) of M containing p so that $U \cap S$ is a level set of a submersion $\Phi : U \to \mathbb{R}^{m-k}$.*

Proof. This is an easy consequence of k-slice condition and submersion level set corollary. □

Example 3.10 (Surface of Revolutions)**.**

4 Immersed Submanifolds

A more general class of submanifolds is the *immersed submanifolds*. It is a submanifold with its own topology so that the inclusion is a smooth immersion. Like in embedded submanifolds, they are precisely images of injective smooth immersions. The figure-eight and the dense curves on torus are both examples of immersed submanifolds. Since immersions are locally embeddings, we see that immersed submanifolds are locally embedded submanifolds.

5 Tangent Space to a Submanifold

Let $S \subset M$ be a submanifold, embedded or immersed, the inclusion $\iota : S \hookrightarrow M$ induces an injective linear map $d\iota_p : T_pS \to T_pM$. Therefore, we may view the tangent space to a submanifold as a vector subspace of the tangent space of the ambient space. We describe a few characterizations of these subspaces. They provide useful ways to describe the tangent space of abstract manifolds: by immersing or embedding into a manifold with obvious tangent space.

An obvious one is:

Proposition 5.1. *$v \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma : J \to M$ with $\gamma(J) \subset S$, smooth in S such that $\gamma(0) = p$ and $\gamma'(0) = v$.*

This is simply because any tangent space is the derivative of some curve.

For embedded submanifolds, the definition can be much more explicit.

Proposition 5.2. *Let $S \subset M$ be an embedded submanifold, $p \in S$, then*

$$T_pS \simeq \{v \in T_pM \mid vf = 0 \; \forall f \in C^\infty(M) \text{ such that } f|_S = 0\}.$$

The previous description is more explicit in terms of local defining function.

Proposition 5.3. *Let $S \subset M$ be an embedded submanifold and $\Phi : U \to N$ is a local defining function, then $T_pS = Ker d\Phi_p$.*

This proposition makes perfect calculus sense since S is precisely the set on which Φ is constant, which should has zero derivative.