Overview of Differential Geometry

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1 Prologue

Every standard curriculum for a math degree (pure or applied) starts with two basic courses: advanced calculus and linear algebra. All other mainstream courses (e.g. differential equation, geometry, analysis.. etc) are taught and learned on these bases. As a math student, one should be familiar by now that every definition and thorem in calculus and linear algebra must take place in a set with certain structure, also known as a *space*. It is also well known that in order for calculus to be discussed, the space must be *complete* enough so that the notion of *limit* is well defined. The minimum space for that requirement is the real numbers $\mathbb{R}$, or a tuple of them $\mathbb{R}^n$. In certain instances, algebraic completeness is desired, and we extend the space to the complex numbers $\mathbb{C}$ or $\mathbb{C}^n$. These are known as the *Euclidean spaces*. Advanced calculus and linear algebra, as well as most other undergraduate math courses take place in these spaces (or subsets of them).

However, Euclidean spaces are far too simple. All their important topological invariants are trivial (they are in fact topologically the same as single points). They are insufficient not just for the development of pure math, but are also too restricted for real world applications. We want to *re-define* advanced calculus and linear algebra on more abstract *topological spaces*. However, we do not want to be too abstract so we can still differentiate, integrate, solve differential equations, or measure distances, ... etc. Consequentially, these spaces must not be too different from Euclidean spaces in order to do calculus. These "generalized spaces" we are about to defined are called *manifolds*. The theory that takes place on manifolds are called *differential geometry*, on which we develop basic math again and open the door for advanced research world in mathematics. One of the most exciting novelty in differential geometry is that an enormous amount of analytical and algebraic properties are intimately related (even equivalent) to the geometry and topology of the spaces. These were not explicitly observed (or were subtly hidden) in Euclidean spaces due to their topological trivialities.

2 Local vs. Global

In calculus, we have seen spaces that can not be deformed into Euclidean space without breaking or gluing. Classical examples are circle $\mathbb{S}^1$, or sphere $\mathbb{S}^2$:

Example 2.1.

Nevertheless, all the familiar calculus objects such as tangent lines (planes), arclength, surface area, ...etc all make sense here. In fact, recall the definition of $f'(p)$ for a function in elementary calculus:

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}.$$

The formula above, which takes a limit as $h \to 0$, only needs to know the function on *any* neighborhood of p. That is, we do not need to know f on points any positive distance away from p. More precisely, $f'(p)$ is a *local* definition at p and we do not need to study the function on its entire domain.

Take this local concept to Example 2.1. For any point on our spaces, if we do not look too far, the spaces *are* can be identified with Euclidean spaces $\mathbb{R}$ and $\mathbb{R}^2$ smoothly (these are called *coordinates*):

These spaces can therefore be covered some open sets that can be identified with $\mathbb{R}^n$ and differentiations can therefore be defined at least in principle (and hence tangent lines, planes, or even more advanced theorems like inverse function theorem.)

However, when things are defined locally, we are always concerned with well-definedness. As can be seen above, every point can have more than one Euclidean neighborhood and different choice *can* give rise to different expressions of a function and its derivative. It was not a problem in calculus since the spaces we define our functions are mostly $\mathbb{R}^n$ itself (or its open subsets) and only one coordinate works for the entire domain. Coordinates are *global* in elementary calculus courses.

The possible lack of global-ness have many impacts on our old knowledge (and actually make things much more interesting). Perhaps the most immediate ones are

1. What do we mean by a *differentiable function* $f : M \to N$ if derivatives depend on choices of coordinates?
2. How do we *integrate* a function defined on a manifold?
3. Given some definitions defined locally, how do we construct a global definition that agree with the local ones?

All three questions above deal with how we change coordinate from one to another. Equivalently, differential geometry starts with a bunch of Euclidean spaces that are *glued* together via *transition maps.* The ways we glue these local coordinated determine the geometry of the manifolds.

3 Planned Topics

In this semester, we certainly will spend considerable time on fundamental concepts above. Then we will re-discuss knowledge we are familiar with in advanced calculus. These include

1. Tangent spaces of a manifold.

2. The pushforwards and pullbacks.

3. Submersion, immersion, embedding and the concepts of submanifolds.

4. Tensor and differential forms.

5. The orientation of a manifold.

6. Integration and the Stoke's Theorem.

Enjoy!