Name and Student ID's: ____________________

Homework 4, Advanced Calculus 1

1. Consider the set

$$X := \{f : [0,1] \to \mathbb{R} \mid f \quad \textit{continuous}\}.$$

   Prove that the following two functions are metrics on X:

   (a)

$$d(f,g) = \sup_{x\in[0,1]} \{|f(x) - g(x)|\}.$$

   (b)

$$d(f,g) = \left(\int_0^1 |f-g|^p \, dx \right)^{\frac{1}{p}}, \ \textit{for } p \in [1,\infty).$$

2. All the definitions defined on $\mathbb{R}^n$ are defined identically on a metric space (X,d). For boundedness, a metric space (X,d), a subset $E \subset X$ is *bounded* if it is contained in some ball. With these, prove that a discrete metric space (X,d) (with $d(x,y) = 0$ if $x = y$ and $d(x,y) = 1$ if $x \neq y$) with more than 1 element is bounded and disconnected. For any subset E of X, what is $\overline{E}$ and $int(E)$?

3. Let d_1 and d_2 be two metrics on a set X such that for some $\alpha, \beta > 0$,

$$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y),$$

   for all $x, y \in X$. Prove that a subset $E \subset X$ is open with respect to d_1 if and only if it is open with respect to d_2. That is, d_1 and d_2 determine the same open subsets. (Such metrics are called *strongly equivalent* and the conditions to be proved are called *topologically equivalent*.)

4. Bartle, Section 14 (pp. 97-98) Exercise B.

5. Bartle, Section 14 (pp. 97-98) Exercise F.

6. Bartle, Section 14 (pp. 97-98) Exercise I.

7. Bartle, Section 14 (pp. 97-98) Exercise J.

8. Bartle, Section 14 (pp. 97-98) Exercise L.