



The Kapustin–Witten equations and nonabelian Hodge theory

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Abstract

Arising from a topological twist of $\mathcal{N} = 4$ super Yang–Mills theory are the Kapustin–Witten equations, a family of gauge-theoretic equations on a four-manifold parametrised by $t \in \mathbb{P}^1$. The parameter corresponds to a linear combination of two super charges in the twist. When $t = 0$ and the four-manifold is a compact Kähler surface, the equations become the Simpson equations, which was originally studied by Hitchin on a compact Riemann surface, as demonstrated independently in works of Nakajima and the third-named author. At the same time, there is a notion of λ -connection in the nonabelian Hodge theory of Donaldson–Corlette–Hitchin–Simpson in which λ is also valued in \mathbb{P}^1 . Varying λ interpolates between the moduli space of semistable Higgs sheaves with vanishing Chern classes on a smooth projective variety (at $\lambda = 0$) and the moduli space of semisimple local systems on the same variety (at $\lambda = 1$) in the twistor space. In this article, we utilise the correspondence furnished by nonabelian Hodge theory to describe a relation between the moduli spaces of solutions to the equations by Kapustin and Witten at $t = 0$ and $t \in \mathbb{R} \setminus \{0\}$ on a smooth, compact Kähler surface. We then provide supporting evidence for a more general form of this

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relation on a smooth, closed four-manifold by computing its expected dimension of the moduli space for each of $t = 0$ and $t \in \mathbb{R} \setminus \{0\}$.

Keywords Kapustin–Witten theory · Nonabelian Hodge theory · λ -connection · Closed four-manifold · Higgs bundle · Flat bundle · Harmonic bundle · Hermitian–Yang–Mills metric · Moduli space · Kähler geometry

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1 Introduction

In this article, we consider a family of equations introduced by Kapustin and Witten [32] on a closed four-manifold with the aim of describing their relationship to a higher-dimensional instance of the nonabelian Hodge correspondence [44, 46]. To set the stage for it, we let G be a compact Lie group and X a closed, oriented, smooth four-manifold, with $P \rightarrow X$ a principal G -bundle over X . We fix a Riemannian metric on X . Furthermore, let A be a connection on P , and let \mathfrak{a} be a smooth section of the bundle $\text{ad}(P) \otimes \Lambda_X^1$, where $\Lambda_X^1 := T^*X$. We say that such data (A, \mathfrak{a}) is solution to the (G) -Kapustin–Witten equations on X for $t \in \mathbb{P}^1$ if they satisfy

$$(F_A - [\mathfrak{a} \wedge \mathfrak{a}] + t d_A \mathfrak{a})^+ = 0, \quad (1.1)$$

$$(F_A - [\mathfrak{a} \wedge \mathfrak{a}] - t^{-1} d_A \mathfrak{a})^- = 0, \quad (1.2)$$

$$d_A^* \mathfrak{a} = 0, \quad (1.3)$$

where \pm in the first two conditions are the projections to the self-dual and anti-self-dual parts of $\text{ad}(P)$ -valued two-forms respectively, F_A is the usual curvature of A , and d_A^* is the formal adjoint of the covariant derivative d_A . Here, t is a parameter that indexes the family of equations (1.1)–(1.3). Since the equations (1.1)–(1.3) form an overdetermined system for $t \notin \mathbb{R} \cup \{\infty\}$ and the equations for $t = \infty$ are the orientation-reversed equations for $t = 0$, we restrict ourselves to the case $t \in \mathbb{R}$ in this article.

Taubes [52] examined the asymptotic behaviour of the system of the equations (1.1)–(1.3) for $t = 0$ as a means of compactifying the moduli space of solutions. One may ask about the algebro-geometric counterpart of this construction, as it may yield a different but definitely useful approach to the problem as in Donaldson’s theory or other gauge-theoretic moduli problems in even dimensions. To facilitate this question, it is appropriate to let X be a compact Kähler surface. In this case, the $t = 0$ equations reduce to the Simpson equations (cf. [39, 50]), which are a generalisation of the Hitchin equations [25]. Thus, the moduli space for $t = 0$ is, algebro-geometrically, that of semistable (or polystable) Higgs sheaves by a result of Simpson [43].

The moduli space of semistable Higgs sheaves is one of the main objects in non-abelian Hodge theory. The other one is the moduli of semisimple local systems, that is, the space of representations of the fundamental group $\pi_1(X)$ via the Riemann–Hilbert

correspondence (see e.g. [22]). The classical nonabelian Hodge theorem [44, 46–48] states that the moduli space of semistable Higgs sheaves with vanishing Chern classes and that of semisimple local systems are real-analytically isomorphic.

Furthermore, this can be realised by means of λ -connections, which are connections parametrized by \mathbb{P}^1 that interpolate between these two moduli spaces at $\lambda = 0$ and $\lambda = 1$, respectively, via the twistor space (see [44, 47–49]). The interpolation can be identified with hyperkähler rotation, as the distinct complex structures that endow the underlying smooth manifold with the structure of the moduli space of semistable Higgs sheaves with vanishing Chern classes and that of semisimple local systems, respectively, are two complex structures in a hyperkähler triple. These structures are two points on a copy of S^2 , which can be identified with the twistor line coordinatized by λ (cf. [22, 44] for more on this point of view).

One may observe perhaps the situation in the Kapustin–Witten theory is similar to that of this nonabelian Hodge theory, especially on a compact Kähler surface, where the semistable Higgs moduli space is located at the origin of the parameter space \mathbb{P}^1 as mentioned above. Then one may wonder what the shape of the Kapustin–Witten equations on a compact Kähler surface for $t \neq 0$ would be and whether it could be related to nonabelian Hodge theory. In this article, we give answers to the questions. To describe them, let X be a smooth, compact Kähler surface, and let E be a Hermitian vector bundle on X with Hermitian metric k . We denote by $\mathcal{A}_{(E,k)}$ the space of k -unitary connections on E and by $\text{End}(E)_0 = \text{End}(E, k)_0$ the bundle of trace-free skew-Hermitian endomorphisms of E on X . We decompose $T^*X \otimes \mathbb{C}$ into $\Lambda_X^{1,0} \oplus \Lambda_X^{0,1}$ by the complex structure of X . In Sect. 4 of this article, we prove the following:

Proposition 4.3 *Let X and E be as above. Suppose $c_1(E) = c_2(E) = 0$. Then, the Kapustin–Witten equations for $t \in \mathbb{R} \setminus \{0\}$ reduce to the following equations on X for $(A, \phi) \in \mathcal{A}_{(E,k)} \times \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$:*

$$F_A^{0,2} = 0, \quad \bar{\partial}_A \phi = 0, \quad \phi \wedge \phi = 0, \quad (1.4)$$

$$F_A^{1,1} + [\phi \wedge \phi^*] = 0, \quad \partial_A \phi = 0. \quad (1.5)$$

Note that these are independent of t as originally observed by Gagliardo and Uhlenbeck for closed four-manifolds [20].

Surprisingly, the system (1.4), (1.5) is equivalent to the one defining *pluri-harmonic metrics* on a Higgs bundle on X in nonabelian Hodge theory — our Sect. 3 includes a brief account on pluri-harmonic metrics in the context of nonabelian Hodge theory. Consequently, the moduli space of solutions to (1.4), (1.5) can be naturally identified with that of semisimple local systems. Moreover, via nonabelian Hodge theory, we obtain the following:

Theorem 4.7 *Let X be a smooth, compact Kähler surface X , and let E be a Hermitian vector bundle on X with $c_1(E) = c_2(E) = 0$. Then the moduli space of solutions to the $t = 0$ Kapustin–Witten equations (4.1), (4.2) on E over X is real-analytically isomorphic (away from any singular loci) to the moduli space of solutions to the $t \neq 0$ Kapustin–Witten equations (4.3), (4.4) on E over X .*

We expect that there would be a similar correspondence between the moduli spaces on a compact smooth four-manifold as well (in the absence of the Kähler geometry). However, there are a number of technical issues that must be overcome in the more general case, such as issues of smoothness and transversality of the moduli spaces. That said, we do prove:

Corollary 5.3 *Consider a principal G -bundle P over a closed, oriented, smooth four-manifold X , where G is $SU(2)$ or $SO(3)$. Then the expected dimensions of moduli spaces of solutions to the Kapustin–Witten equations for $t = 0$ and $t \neq 0$ over X coincide under a particular flatness condition (the vanishing of $k(G)$, as defined in Definition 2.2).*

This result can be taken as evidence in support of a more general correspondence, and it is then definitely interesting to study deformation spaces of solutions to the Kapustin–Witten equations for both $t = 0$ and $t \neq 0$ on a closed four-manifold in an appropriate sense; and to compare the moduli spaces of them to obtain a certain form of correspondence indicated by nonabelian Hodge theory. A promising language for this is available in the derived geometry of Joyce [29, 30], given that the objects are topological in nature and so the standard metric perturbation method in gauge theory would not work nicely for them. Note that Pantev–Toën–Vaquié–Vezzosi [40] gives a (-2) -shifted symplectic structure to the moduli stack of perfect complexes of local systems on a compact, oriented, topological four-manifold X , which lives in the $t \neq 0$ side of the Kapustin–Witten theory. The orientation problem for the moduli space of solutions to the Kapustin–Witten equations for $t = 0$ was solved in [31, Theorem 4.9], and it is just straightforward to see that the same statement holds for the $t \neq 0$ case.

Lastly, one can study the G -Kapustin–Witten equations for the case of reductive, non-compact groups $G_{\mathbb{C}}$ such as $G_{\mathbb{C}} = GL(r, \mathbb{C})$, $SL(r, \mathbb{C})$, and $PGL(r, \mathbb{C})$. Given what is observed for Hitchin systems on curves [11, 24, 32], it is compelling to ask whether moduli spaces of Kapustin–Witten solutions admit a fibration, in the sense of the Hitchin fibration [26] (cf. [9, 48] for higher dimensions), that respects Langlands duality. In particular, one might compare the moduli spaces for $G_{\mathbb{C}} = SL(r, \mathbb{C})$ moduli space and ${}^L G_{\mathbb{C}} = PGL(r, \mathbb{C})$ and ask whether they are “dual” with respect to this fibration. Related to this hypothesis is the more general question of whether the moduli spaces Calabi–Yau in some appropriate sense (or even hyperkähler), in which case the duality may have a physical interpretation (although we do not speculate on this). To answer these questions, one must first define the appropriate fibration on the total spaces carefully. We leave this for future work.

To summarize, the structure of this article is as follows:

- Section 2 describes basic properties of solutions to the Kapustin–Witten equations on closed four-manifolds.
- Section 3 is a brief overview of classical nonabelian Hodge theory with a view to the application we have in mind.
- We study the Kapustin–Witten equations on a compact Kähler surface and describe their relationship to the nonabelian Hodge theory in Sect. 4.
- In Sect. 5, we compute the expected dimensions of the moduli spaces of solutions to both the $t = 0$ and $t \neq 0$ Kapustin–Witten equations on a general smooth, closed four-manifold.

2 A family of equations in Kapustin–Witten theory

Let X be a closed, oriented, smooth four-manifold, and let $P \rightarrow X$ be a principal G -bundle over X , where G is a compact Lie group. Fix a Riemannian metric on X . It induces the Hodge star operator $*$: $\Lambda^p T^*X \rightarrow \Lambda^{4-p} T^*X$ for $p = 0, \dots, 4$. Since $*^2 = 1$ on $\Lambda^2_X := \Lambda^2 T^*X$, Λ^2_X decomposes into *self-dual* part $\Lambda^2_X^+$ and *anti-self-dual* part $\Lambda^2_X^-$, corresponding to the eigenvalues ± 1 respectively. Accordingly, we have the decomposition $\text{ad}(P) \otimes \Lambda^2_X = \text{ad}(P) \otimes \Lambda^2_X^+ \oplus \text{ad}(P) \otimes \Lambda^2_X^-$.

From a topological twist of $\mathcal{N} = 4$ super Yang–Mills theory, Kapustin and Witten [32] introduced the following family of equations parametrized by $t \in \mathbb{P}^1$ for a pair (A, \mathfrak{a}) consisting of a connection A on P and a section \mathfrak{a} of the bundle $\text{ad}(P) \otimes \Lambda^1_X$:

$$(F_A - [\mathfrak{a} \wedge \mathfrak{a}] + t d_A \mathfrak{a})^+ = 0, \tag{2.1}$$

$$(F_A - [\mathfrak{a} \wedge \mathfrak{a}] - t^{-1} d_A \mathfrak{a})^- = 0, \tag{2.2}$$

$$d_A^* \mathfrak{a} = 0, \tag{2.3}$$

where the superscripts \pm denote the respective projections to the self-dual part $\Gamma(\text{ad}(P) \otimes \Lambda^1_X^+)$ and the anti-self-dual one $\Gamma(\text{ad}(P) \otimes \Lambda^1_X^-)$ respectively, and d_A^* is the formal adjoint of the covariant derivative d_A with respect to the L^2 -inner product. As mentioned in Introduction, we restrict ourselves to the case $t \in \mathbb{R}$ since the equations are overdetermined for $t \notin \mathbb{R} \cup \{\infty\}$ and the equations for $t = \infty$ is the orientation-reversed one for $t = 0$. For the case $t \in \mathbb{R}$, the equations (2.1)–(2.3) with a gauge fixing equation form an elliptic system.

Remark 2.1 The physical parameter on which the Kapustin–Witten theory depends is a combination of this twisting parameter t and the coupling parameter coming from the coupling constant and the theta-angle in the theory. See [32, Section 3.5] for more details.

Kapustin and Witten [32] discovered an astonishing relationship between the geometric Langlands programme on a curve and the S -duality conjecture for $\mathcal{N} = 4$ super Yang–Mills theory in dimension four via the topologically-twisted theory compactified on the curve, which was further generalised to the ramified case by Gukov–Witten [23]. It is worth noting that this relationship interacts with the hyperkähler structure on the moduli space of semistable Higgs bundles over the curve. This interaction picks out distinguished triple branes (branes that are either symplectic or complex with regards to each complex structure) in the moduli space that are transformed under mirror symmetry, which itself is a form of global Langlands duality (see also Hausel–Thaddeus [24] and Donagi–Pantev [11]). The classification of triple branes in semistable Higgs bundle moduli spaces is explored in [2, 3, 5, 18] for instance, and has been furthermore linked to Nakajima quiver varieties in [19, 41]. There are also emerging connections between Kapustin–Witten theory and multiplicative Hitchin systems that are explored in [16].

In this article, however, the underlying four-manifold is not necessarily the product of two Riemann surfaces, and we describe nonabelian-Hodge-theoretic aspect of the

theory on complex surfaces, rather than on curves, and discuss a further possible realisation of it on closed four-manifolds.

For $t = 0$ the equations (2.1)–(2.3) become

$$F_A^+ - [\mathfrak{a} \wedge \mathfrak{a}]^+ = 0, \quad (d_A \mathfrak{a})^- = 0, \quad d_A^* \mathfrak{a} = 0. \quad (2.4)$$

Solutions to the above equations (2.4), especially their asymptotic behaviours, were intensively studied by Taubes [52].

We now require the following definition:

Definition 2.2 For a principal G -bundle P over X , we define the integer $k(G)$ to be $c_2(P \otimes \mathbb{C})[X]$ if G is $SU(2)$ and by $-\frac{1}{4}p_1(P)[X]$ if G is $SO(3)$.

Gagliardo and Uhlenbeck [20, Corollary 3.3] show that the equations (2.1)–(2.3) have a solution if and only if $k(G) = 0$, and they are independent of t for $t \in \mathbb{R} \setminus \{0\}$. Namely, they prove:

Theorem 2.3 (Gagliardo–Uhlenbeck) *Assume that the structure group G is $SU(2)$ or $SO(3)$. Then the equations (2.1)–(2.3) have a solution if and only if $k(G) = 0$. Furthermore, a pair (A, \mathfrak{a}) satisfies the equations (2.1)–(2.3) with $t \in \mathbb{R} \setminus \{0\}$ if and only if it satisfies the following equations:*

$$F_A - [\mathfrak{a} \wedge \mathfrak{a}] = 0, \quad d_A \mathfrak{a} = d_A^* \mathfrak{a} = 0. \quad (2.5)$$

The clever trick used by Gagliardo and Uhlenbeck to prove the above theorem is to consider the curvature $F_{A+i\mathfrak{a}}$ of the “complexified” connection $A + i\mathfrak{a}$ out of a pair (A, \mathfrak{a}) which satisfies the Kapustin–Witten equations (2.1)–(2.3) with $t \in \mathbb{R} \setminus \{0\}$ and the use of their fascinating energy identity for it (see [20, Section 3] for more details).

Similar equations to (2.5) on smooth three-manifolds were studied by Taubes [51] in the process of analytically describing the *Morgan–Shalen compactification* of $SL(2, \mathbb{C})$ -character varieties of three-manifolds.

Remark 2.4 The above t -independency is not true when X is not compact (see e.g. Gaiotto–Witten [21] which develops the programme by Witten [55] aiming at “categorifying” the knot invariants by using the system of the equations (2.1)–(2.3) on a four-manifold with boundary).

3 Nonabelian Hodge theory

For general references on classical nonabelian Hodge theory, we refer the reader to [22, 27, 37, 44, 46, 48, 49]. In order to employ the necessary elements of the correspondence, we will need to introduce some algebro-geometric language. Also, we will take a more general point of view in which X is not necessarily of real dimension four, at least initially. As such, we let X be a smooth, compact Kähler manifold of complex dimension n with Kähler form ω . Let its holomorphic cotangent bundle be denoted by Ω_X^1 and let E be a smooth complex vector bundle on X . We are interested in two types of structures on E : *Higgs structures* and *flat structures*. The

former structure is that of a Higgs bundle while the latter is that of a flat connection. The former necessitates that E is Hermitian while the latter is not necessarily unitary with respect to any Hermitian metric.

3.1 Higgs bundles and Hermitian-Yang–Mills metrics

Suppose now that $\bar{\partial}_E$ is a holomorphic structure on E . Then, a \mathcal{O}_X -linear, $\bar{\partial}_E$ -holomorphic map $\theta: E \rightarrow E \otimes \Omega_X^1$ is called a *Higgs field* for $(E, \bar{\partial}_E)$. When the integrability condition $\theta \wedge \theta = 0$ is satisfied, we say that the triple $(E, \bar{\partial}_E, \theta)$ is a *Higgs bundle* on X .

We define a notion of *degree* for E by $\deg(E) := \int_X c_1(E) \cdot \omega^{n-1}$. Following from this, the *slope* of E is

$$\mu(E) := \deg(E)/\text{rank}(E).$$

We will use \mathcal{E} to refer to the locally-free sheaf of holomorphic sections of $(E, \bar{\partial}_E)$. We define the rank, degree, and slope of the Higgs bundle $(E, \bar{\partial}_E, \theta)$ to be those of the underlying vector bundle E (or sheaf \mathcal{E}), respectively. With these ideas in place, we can define *stability*:

Definition 3.1 We say $(E, \bar{\partial}_E, \theta)$ is *semistable* if, for each nonzero coherent subsheaf $\mathcal{F} \subsetneq \mathcal{E}$ with the property $\theta(\mathcal{F}) \subseteq \mathcal{F} \otimes \Omega_X^1$ (i.e. with that property that it is θ -invariant), then we have

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}).$$

We call $(E, \bar{\partial}_E, \theta)$ *stable* if we replace \leq with $<$ in the preceding sentence.

Definition 3.2 We call $(E, \bar{\partial}_E, \theta)$ *polystable* if it is a direct sum of stable Higgs bundles of the same slope.

Remark 3.3 One needs to consider Gieseker-semistable Higgs sheaves in order to set up the moduli problem properly on a surface or a higher-dimensional variety. However, we do not do so, since our focus here is on the correspondence between analytic objects and algebro-geometric ones. We restrict attention in this article to bundles (locally-free sheaves) rather than more general coherent sheaves because (a) the Kapustin–Witten equations are naturally written on a smooth bundle and (b) the ensuing discussion of the existence of special metrics is most natural within the setting of bundles.

Now, given a Higgs bundle $(E, \bar{\partial}_E, \theta)$ on X , consider a Hermitian metric h on E . The combination of this data induces a $(1, 0)$ -connection $\partial_{E,h}$ so that $D_{\partial_{E,h}}^1 := \bar{\partial}_E + \partial_{E,h}$ is a unique h -unitary connection on E compatible with $\bar{\partial}_E$, which is called the *Chern connection* associated to $(\bar{\partial}_E, h)$. Denote by $\theta^{*,h}$ the adjoint of the Higgs field with respect to the Hermitian metric h on E .

Definition 3.4 A Hermitian metric h on a Higgs bundle $(E, \bar{\partial}_E, \theta)$ is said to be a *Hermitian-Yang–Mills* one if $\Lambda(F_h + [\theta, \theta^{*,h}]) = c \text{Id}_E$, where Λ is the L^2 -adjoint of $\wedge \omega$, F_h is the curvature of $D_{\partial_{E,h}}^1$ and a constant c is given by $-\frac{2\pi i}{r(n-1)! \int_X \omega^n} \cdot \deg(E)$.

Simpson [43] proves the following by using the heat equation method of Uhlenbeck–Yau in the stable bundle case [53].

Theorem 3.5 (Simpson) *A Higgs bundle admits a Hermitian–Yang–Mills metric if and only if it is polystable.*

3.2 Flat bundles and harmonic metrics

The other primary object in nonabelian Hodge theory is a *flat bundle* on X , namely, a pair (E, ∇) consisting of the smooth complex vector bundle E and a flat $\mathrm{GL}(r, \mathbb{C})$ -connection ∇ on E , where r is the rank of E . This is equivalent to considering a representation of the fundamental group of X in $\mathrm{GL}(r, \mathbb{C})$ via the Riemann–Hilbert correspondence. We say (E, ∇) is *irreducible* if E has no non-trivial ∇ -invariant subbundle.

Definition 3.6 We call (E, ∇) *semisimple* if it is a direct sum of irreducible flat bundles.

Fix $x \in X$. Then a flat bundle (E, ∇) gives a monodromy representation $\rho: \pi_1(X, x) \rightarrow \mathrm{GL}(E|_x, \mathbb{C})$. Denote by $\pi: \tilde{X} \rightarrow X$ the universal cover of X . We fix a point $\tilde{x} \in \tilde{X}$ with $\pi(\tilde{x}) = x$. We consider a Hermitian metric on E and denote it by h . Then it gives a ρ -equivariant map $u_h: \tilde{X} \rightarrow \mathrm{GL}(n, \mathbb{C})/U(n)$.

Definition 3.7 A Hermitian metric h on E is said to be a *harmonic metric* if the induced ρ -equivariant map $u_h: \tilde{X} \rightarrow \mathrm{GL}(n, \mathbb{C})/U(n)$ is a harmonic one.

The following is attributed to Corlette [10], Donaldson [12], Jost–Yau [28] and Labourie [34], it is proved by the heat equation method pioneered by Eells–Sampson [15].

Theorem 3.8 *A flat bundle (E, ∇) admits a harmonic metric if and only if it is semisimple.*

3.3 Harmonic bundles

Let $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle on X . Consider a Hermitian metric h on E . This induces a $(1, 0)$ -connection $\partial_{E,h}$ so that $D_{\bar{\partial}_E, h}^1 := \bar{\partial}_E + \partial_{E,h}$ is a unique h -unitary connection on E compatible with $\bar{\partial}_E$. This also defines $\theta^{*,h} \in \Omega^{0,1}(\mathrm{End}(E))$ as above. Set $\mathbb{D}_h^1 := D_{\bar{\partial}_E, h}^1 + \theta + \theta^{*,h}$, this is a connection on E .

Definition 3.9 Let $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle. A Hermitian metric h on E is said to be a *pluri-harmonic metric* on $(E, \bar{\partial}_E, \theta)$, if \mathbb{D}_h^1 is flat, i.e. $\mathbb{D}_h^1 \circ \mathbb{D}_h^1 = 0$. We call such a quadruplet $(E, \bar{\partial}_E, \theta, h)$ a *harmonic bundle*.

On the other hand, the notion of pluri-harmonic metrics and harmonic bundles are defined from a flat bundle. Let (E, ∇) be a vector bundle E on X with flat connection ∇ . Suppose we are given a Hermitian metric h on E . Then ∇ uniquely decomposes into $D_h^0 + \Theta_h$, where D_h^0 is a h -unitary connection and Θ_h is the self-adjoint part. Decompose this D_h^0 into the $(1,0)$ and $(0,1)$ parts as $D_h^0 = \partial_h + \bar{\partial}_h$. Define θ_h and θ_h^* as the $(1,0)$ and $(0,1)$ parts of Θ_h .

Definition 3.10 We call a Hermitian metric h on E a *pluri-harmonic metric* on (E, ∇) , if $\mathbb{D}_h^0 := \bar{\partial}_h + \theta_h$ satisfies $\mathbb{D}_h^0 \circ \mathbb{D}_h^0 = 0$, namely, $(E, \bar{\partial}_E, \theta_h)$ is a Higgs bundle. We also say (E, ∇, h) a *harmonic bundle*.

The object \mathbb{D}_h^1 introduced in the above is a connection (not necessarily flat) constructed from a Higgs bundle, while the object \mathbb{D}_h^0 is a Higgs bundle structure (not necessarily integrable) constructed from a flat connection. The question is whether we can recover one type of structure from the other and when these structures are flat (respectively, integrable). The answers are in fact related. When we start with a Higgs bundle $(E, \bar{\partial}_E, \theta)$ for which $\theta \wedge \theta = 0$, and if there exists a Hermitian metric h such that $\mathbb{D}_h^1 \circ \mathbb{D}_h^1 = 0$, then constructing \mathbb{D}_h^0 from \mathbb{D}_h^1 recovers the Higgs bundle $(E, \bar{\partial}_E, \theta)$ and $\mathbb{D}_h^1 \circ \mathbb{D}_h^1 = 0$. Conversely, when we start with a flat connection (E, ∇) for which $\nabla^2 = 0$, and if there exists a Hermitian metric h such that $\mathbb{D}_h^0 \circ \mathbb{D}_h^0 = 0$, then constructing \mathbb{D}_h^1 from \mathbb{D}_h^0 recovers (E, ∇) and $\mathbb{D}_h^0 \circ \mathbb{D}_h^0 = 0$. Thus we are searching for the existence of a Hermitian metric h that intertwines Higgs and flat structures in this way.

3.4 Nonabelian Hodge correspondence

We briefly recall the notion of λ -connection introduced by Deligne and Simpson. It interpolates the polystable Higgs bundles with vanishing Chern classes and semisimple flat ones via the twistor space intrinsically attached to them (see e.g. [44, 49] for more details).

Definition 3.11 Let $\lambda \in \mathbb{C}$. A complex linear operator $\mathbb{D}^\lambda : E \rightarrow E \otimes \Omega_X^1$ is said to be a λ -connection if it satisfies

$$\mathbb{D}^\lambda(f \cdot s) = f\mathbb{D}^\lambda(s) + \lambda s \otimes df,$$

where f is a function of X and s is a section of E . This extends to $\mathbb{D}^\lambda : E \otimes \Omega_X^p \rightarrow E \otimes \Omega_X^{p+1}$. It is said to be *flat* if $\mathbb{D}^\lambda \circ \mathbb{D}^\lambda = 0$. We call a pair (E, \mathbb{D}^λ) with flat λ -connection \mathbb{D}^λ a λ -flat bundle. We say it is *semisimple* if it is a direct sum of irreducible λ -flat bundles.

Remark 3.12 Note that the data in Definition 3.11, while a priori a smooth λ -connection, will in fact be equivalent to a *holomorphic* λ -connection in our development due to the assumption of compatibility with a Hermitian metric on E .

For $\lambda = 1$ this object is a connection in usual sense; for $\lambda = 0$, it becomes a Higgs field. Considered as a family over \mathbb{C} , this construction smoothly transforms a Higgs bundle with vanishing Chern classes into a flat bundle. More precisely, consider a Hermitian metric h on E with flat λ -connection \mathbb{D}^λ as above. Then the λ -connection uniquely decomposes into $\lambda \bar{\partial}_{E,h} + \theta_h + \bar{\partial}_{E,h} + \lambda \theta_h^{*,h}$, where θ_h is a section of $\text{End}(E) \otimes \Lambda_X^{1,0}$ and $\theta_h^{*,h}$ is that of $\text{End}(E) \otimes \Lambda_X^{0,1}$ (see e.g. [37, Section 2]). Thus, \mathbb{D}^λ with $\lambda = 1$ is exactly the \mathbb{D}_h^1 before, and so does the $\lambda = 0$ case. Moreover, one can naturally form a family of moduli spaces on \mathbb{P}^1 , where the central fibre

is the moduli space of polystable Higgs bundles with vanishing Chern classes and the other fibres are those of semisimple λ -flat bundles. This is known usually as the *Deligne–Hitchin moduli space*.

We now state the classical nonabelian Hodge correspondence by Simpson [44, 46, 48, 49] in the following form:

Theorem 3.13 *For each $\lambda \in \mathbb{C}$, there is a real analytic isomorphism (away from any singular loci) between the moduli spaces of polystable Higgs bundles with vanishing Chern classes and semisimple λ -flat bundles on a smooth, compact Kähler manifold.*

The correspondence has been generalised to non-compact cases in a series of works, including those of Simpson [45] for the tame case and of Biquard–Boalch [4] for the wild case, both in complex dimension one. Mochizuki has extended both the tame and wild cases to arbitrary dimension [36–38], while nonabelian Hodge theory has been developed in a number of further directions (e.g. [17] for cuspidal curves in a modular context). We refer to a recent survey [27] and its various references for the state of these developments.

4 The equations on compact Kähler surfaces

In this section, we describe the equations (2.4) (the Kapustin–Witten equations for $t = 0$) and (2.5) (the Kapustin–Witten equations for $t \neq 0$) on a compact Kähler surface. It turns out that they become familiar equations in nonabelian Hodge theory described in Sect. 3.

Let X be a compact Kähler surface X with Kähler form ω , and let E be a Hermitian vector bundle of rank r on X with Hermitian metric k . We assume $c_1(E) = 0$ throughout this section. We denote by $\mathcal{A}_{(E,k)}$ the space of k -unitary connections on E and by $\text{End}(E)_0 = \text{End}(E, k)_0$ the bundle of trace-free skew-Hermitian endomorphisms of E on X . We decompose $T^*X \otimes \mathbb{C}$ into $\Lambda_X^{1,0} \oplus \Lambda_X^{0,1}$ as well as the covariant derivative d_A associated to $A \in \mathcal{A}_{(E,k)}$ into $\partial_A + \bar{\partial}_A$ by the complex structure of X ; and write $\alpha = \phi - \phi^*$, where $\phi \in \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$, and $\phi^* \in \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{0,1})$ is the adjoint of ϕ with respect to k . Note that $\Lambda_X^{1,0}$ is the same as the holomorphic cotangent bundle Ω_X^1 of X .

4.1 The equations for $t = 0$

For $t = 0$, Nakajima [39, Section 6 (iii)] and the third-named author [50, Proposition 2.5] obtained the following:

Proposition 4.1 *Let E and X be as above. Then, the Kapustin–Witten equations (2.4) for $t = 0$ have the following form on a smooth, compact Kähler surface, searching for $(A, \phi) \in \mathcal{A}_{(E,k)} \times \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$:*

$$F_A^{0,2} = 0, \quad \bar{\partial}_A \phi = 0, \quad \phi \wedge \phi = 0, \quad (4.1)$$

$$\Lambda(F_A^{1,1} + [\phi \wedge \phi^*]) = 0, \quad (4.2)$$

where $\Lambda := (\wedge \omega)^*$.

We then reformulate the system of the above equations (4.1), (4.2) in terms of Hermitian metrics on E instead of connections of it to clarify a link to nonabelian Hodge theory described in Sect. 3.

Proposition 4.2 *Let E be a complex vector bundle of rank r with $c_1(E) = 0$ over a smooth, compact Kähler surface X . Fix a Hermitian metric k on E . Then, searching for a solution $(A, \phi) \in \mathcal{A}_{(E,k)} \times \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$ to the system of the equations (4.1), (4.2) on E is equivalent to looking for a Hermitian–Yang–Mills metric on a Higgs bundle $(E, \bar{\partial}_E, \theta)$ with $\text{tr } \theta = 0$ on X . This correspondence is one-to-one up to gauge transformations.*

Proof The proof here is similar to that by Bradlow for the vortex equations in [7, Section 3] and uses some results therein (see also [33, Chapter VI]). First, if $(A, \phi) \in \mathcal{A}_{(E,k)} \times \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$ is a solution to the equation (4.1), we are given a connection A on E with $F_A^{0,2} = 0$, namely, we have a holomorphic structure $\bar{\partial}_A$ defined by A on E . In addition, we have $\phi \in \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$ with $\bar{\partial}_A \phi = 0$ and $\phi \wedge \phi = 0$. Therefore, $(E, \bar{\partial}_A, \phi)$ defines a Higgs bundle with $\text{tr } \phi = 0$ on X . Then, the equation (4.2) implies that k gives a unique Hermitian–Yang–Mills metric on the Higgs bundle $(E, \bar{\partial}_A, \phi)$, since A is the unique k -unitary connection compatible with the holomorphic structure $\bar{\partial}_A$ in this case.

Conversely, suppose that we are given a Higgs bundle $(E, \bar{\partial}_E, \theta)$ with holomorphic structure $\bar{\partial}_E$ and Higgs field θ with $\text{tr } \theta = 0$. We consider a Hermitian metric h on E , which is not necessarily the same as k that we fixed at the beginning. Then, there exists a positive and self-adjoint endomorphism of E with respect to k , we denote it by H , satisfying $h = kH$. This H decomposes uniquely as $H = g^*g$, where g is a complex gauge transformation. Define a holomorphic structure $\bar{\partial}_{E,k} := g(\bar{\partial}_E) = g\bar{\partial}_E g^{-1}$ on E and a section $\phi := g(\theta) = g\theta g^{-1} \in \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$ by the gauge transformation. Then, the tuple $(E, \bar{\partial}_{E,k}, \phi)$ is a Higgs bundle on X , namely, $\bar{\partial}_{E,k} \phi = 0$, $\phi \wedge \phi = 0$. We now consider a unique k -unitary connection $A := D_{\bar{\partial}_{E,k},k}^1 = \bar{\partial}_{E,k} + \partial_{E,k}$ compatible with $\bar{\partial}_{E,k}$ and with $(1, 0)$ -connection $\partial_{E,k}$. We obviously have $F_A^{0,2} = 0$. It is also straightforward to see that $F_A^{1,1} = gF_h^{1,1}g^{-1}$, where $F_h^{1,1}$ is the $(1, 1)$ -part of the curvature F_h of the h -unitary connection $D_{\bar{\partial}_E,h}^1$ compatible with $\bar{\partial}_E$, and $[\phi \wedge \phi^*] = g[\theta \wedge \theta^{*,h}]g^{-1}$ (cf. [7, Lemma 3.4]), namely, we have $\Lambda(F_A^{1,1} + [\phi \wedge \phi^*]) = g(\Lambda(F_h^{1,1} + [\theta \wedge \theta^{*,h}]))g^{-1}$. Hence, (A, ϕ) satisfies the equations (4.1), (4.2) if and only if h is a Hermitian–Yang–Mills metric on the Higgs bundle $(E, \bar{\partial}_E, \theta)$. This construction obviously gives the inverse to the other one at the top of the proof up to gauge transformations. \square

Note that Simpson [43] proves that a Hermitian–Yang–Mills metric on a Higgs bundle exists if and only if the Higgs bundle is polystable in the sense of Definition 3.1 as mentioned also in Sect. 3.1.

4.2 The equations for $t \neq 0$

We first remark that it is straightforward to see that Theorem 2.3 holds for $G = U(r)$ with $c_1(P) = c_2(P) = 0$. Then, for $t \in \mathbb{R} \setminus \{0\}$, we obtain the following:

Proposition 4.3 *Let E be a Hermitian vector bundle of rank r over a smooth, compact Kähler surface X . Suppose that $c_1(E) = c_2(E) = 0$. Then, the Kapustin–Witten equations (2.5) for $t \neq 0$ on X reduce to the following equations for $(A, \phi) \in \mathcal{A}_{(E,k)} \times \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$:*

$$F_A^{0,2} = 0, \quad \bar{\partial}_A \phi = 0, \quad \phi \wedge \phi = 0, \quad (4.3)$$

$$F_A^{1,1} + [\phi \wedge \phi^*] = 0, \quad \partial_A \phi = 0. \quad (4.4)$$

Proof We utilise a method by Nakajima [39, Section 6 (iii)]. Recall first that the following Bochner–Weitzenböck formula for $\mathfrak{a} \in \Gamma(\text{ad}(P) \otimes \Lambda_X^1)$ holds:

$$d_A d_A^* \mathfrak{a} + d_A^* d_A \mathfrak{a} = \nabla_A^* \nabla_A \mathfrak{a} + \text{Ric} \circ \mathfrak{a} + [F_A, \mathfrak{a}], \quad (4.5)$$

where Ric denotes the Ricci transformation of the underlying Riemannian metric of X (see e.g. Bourguignon–Lawson [6, Theorem 3.2] for more details). By taking the L^2 -inner product of the identity (4.5) with \mathfrak{a} , we obtain

$$\begin{aligned} \int_X |d_A^* \mathfrak{a}|^2 + \int_X |d_A \mathfrak{a}|^2 + \frac{1}{2} \int_X |F_A - [\mathfrak{a} \wedge \mathfrak{a}]|^2 \\ = \int_X |\nabla_A \mathfrak{a}|^2 + \frac{1}{2} \int_X |F_A|^2 + \frac{1}{2} \int_X |[\mathfrak{a} \wedge \mathfrak{a}]|^2 + \int_X s |\mathfrak{a}|^2, \end{aligned} \quad (4.6)$$

where s is the scalar curvature of the underlying Riemannian metric on X .

Abusing the notation, we denote $i\phi + i\phi^*$ by $i\mathfrak{a}$ on a compact Kähler surface X . Observe that the left-hand side of (4.6) is zero, if (A, \mathfrak{a}) is a solution to (2.5). On the other hand, the right-hand side of (4.6) is unchanged by replacing \mathfrak{a} by $i\mathfrak{a}$. Hence, if (A, \mathfrak{a}) is a solution to (2.5), so does $(A, i\mathfrak{a})$. In other words, if (A, ϕ) is a solution to (2.5), so does $(A, i\phi)$.

The $(2, 0)$ -part of the first equation in (2.5), namely, $F_A - [\mathfrak{a}, \mathfrak{a}] = 0$, reads $F_A^{2,0} - \phi \wedge \phi = 0$. Since (A, ϕ) and $(A, i\phi)$ are solutions to the equation simultaneously, we obtain $F_A^{2,0} = 0$ and $\phi \wedge \phi = 0$. From the same argument for the $(0, 2)$ -part of the first equation in (2.5), we obtain $F_A^{0,2} = 0$ and $\phi^* \wedge \phi^* = 0$. We obviously obtain $F_A^{1,1} + [\phi \wedge \phi^*] = 0$ from the $(1, 1)$ -part of the first equation in (2.5).

In addition, from the $(2, 0)$ - and $(0, 2)$ -parts of the equation $d_A \mathfrak{a} = 0$, we obtain $\partial_A \phi = \bar{\partial}_A \phi^* = 0$. Since $d_A(i\mathfrak{a}) = 0$, we obtain $\bar{\partial}_A \phi + \partial \phi^* = 0$. From this with the $(1, 1)$ part of the original $d_A \mathfrak{a} = 0$, namely, $\bar{\partial}_A \phi - \partial \phi^* = 0$, we obtain $\bar{\partial}_A \phi = 0$.

For the opposite direction, since $d_A^* \mathfrak{a} = \Lambda(\bar{\partial}_A \phi + \partial_A \phi^*)$ via the Kähler identities (see [50, Section 2.2]), it is automatic to obtain $d_A^* \mathfrak{a} = 0$. \square

Remark 4.4 When the scalar curvature s of the underlying manifold X is positive, the identity (4.6) can be used to provide a *vanishing theorem* for a in the $t \neq 0$ Kapustin–Witten equations (2.5).

Remark 4.5 Ward [54] discusses the system of the equations (4.3), (4.4) as an instance of higher-dimensional completely integrable systems, generalising the notion from real dimension 2 that was initiated by Hitchin [26]. The setting for Ward’s construction is real dimension $2k$ with explicit examples in real dimension 4.

We then reformulate the system of the above equations (4.3), (4.4) in terms of Hermitian metrics on E as before.

Proposition 4.6 *Let E be a complex vector bundle of rank r with over a smooth, compact Kähler surface X . Suppose $c_1(E) = c_2(E) = 0$. Fix a Hermitian metric k on E . Then, searching for a solution $(A, \phi) \in \mathcal{A}_{(E,k)} \times \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$ to the system of the equations (4.3), (4.4) on E is equivalent to looking for a pluri-harmonic metric on a Higgs bundle $(E, \bar{\partial}_E, \theta)$ with $\text{tr } \theta = 0$ on X . This correspondence is one-to-one up to gauge transformations.*

Proof The proof here goes along the same line with that of Proposition 4.2. If $(A, \phi) \in \mathcal{A}_{(E,k)} \times \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$ is a solution to the equations (4.3), we are given a Higgs bundle $(E, \bar{\partial}_A, \phi)$ with $\text{tr } \phi = 0$ as before. Then, (4.4) claims that k is a pluri-harmonic metric on the Higgs bundle $(E, \bar{\partial}_A, \phi)$.

Conversely, start with a Higgs bundle $(E, \bar{\partial}_E, \theta)$ with holomorphic structure $\bar{\partial}_E$ and Higgs field θ . We consider a Hermitian metric h on E , which need not be the same as the fixed Hermitian metric k at the beginning. Then, we have a positive and self-adjoint endomorphism H of E with respect to k , which satisfies $h = kH$. We decompose this H as $H = g^*g$, where g is a complex gauge transformation. Define a holomorphic structure $\bar{\partial}_{E,k} := g(\bar{\partial}_E) = g\bar{\partial}_E g^{-1}$ on E and a section $\phi := g(\theta) = g\theta g^{-1} \in \Gamma(\text{End}(E)_0 \otimes \Lambda_X^{1,0})$ by the gauge transformation. Then, the tuple $(E, \bar{\partial}_{E,k}, \phi)$ is a Higgs bundle on X , that is, $\bar{\partial}_{E,k}\phi = 0, \phi \wedge \phi = 0$. Let us then consider a unique k -unitary connection $A := D_{\bar{\partial}_{E,k},k}^1 = \bar{\partial}_{E,k} + \partial_{E,k}$ compatible with $\bar{\partial}_{E,k}$ and with $(1, 0)$ -connection $\partial_{E,k}$ as before.

On the other hand, we have a unique h -unitary connection $D_{\bar{\partial}_E,h}^1$ compatible with $\bar{\partial}_E$ and with $(1, 0)$ -connection $\partial_{E,h}$. Let us then consider $\mathbb{D}_h^1 := D_{\bar{\partial}_E,h}^1 + \theta + \theta^{*,h}$. This \mathbb{D}_h^1 is a connection which is compatible with the metric h . Since $(E, \bar{\partial}_E, \theta)$ is a Higgs bundle on X , we have $\mathbb{D}_h^1 \circ \mathbb{D}_h^1 = F_{\bar{\partial}_E,h}^{1,1} + [\theta, \theta^{*,h}] + \partial_{E,h}\theta + \bar{\partial}_E\theta^{*,h}$, where $F_h^{1,1}$ is the $(1, 1)$ -part of the curvature F_h of $D_{\bar{\partial}_E,h}^1$. Then, it is just clear that (A, ϕ) satisfies the equations (4.3), (4.4) if and only if $\mathbb{D}_h^1 \circ \mathbb{D}_h^1$ vanishes, namely, h is a pluri-harmonic metric on the Higgs bundle $(E, \bar{\partial}_E, \theta)$, since $F_A^{1,1} + [\phi \wedge \phi^*] = g(F_h^{1,1} + [\theta \wedge \theta^{*,h}])g^{-1}, \partial_{E,k}\phi = g(\partial_{E,h}\theta)g^{-1}$, and $\bar{\partial}_{E,k}\phi^{*,k} = g(\bar{\partial}_E\theta^{*,h})g^{-1}$ (cf. [7, Lemma 3.4]). This construction certainly gives the inverse to the other up to gauge transformations. \square

A pluri-harmonic metric on a Higgs bundle is a Hermitian metric which determines the flat λ -connection at $\lambda = 1$, as described in Sect. 3. Hence, it is natural to think of the moduli space of solutions to the $t \neq 0$ Kapustin–Witten equations (4.3), (4.4)

on a compact Kähler surface as that of semisimple flat bundles. Furthermore, by the nonabelian Hodge correspondence described in Sect. 3, we obtain the following:

Theorem 4.7 *Let X be a smooth, compact Kähler surface X , and let E be a Hermitian vector bundle on X with $c_1(E) = c_2(E) = 0$. Then the moduli space of solutions to the $t = 0$ Kapustin–Witten equations (4.1), (4.2) on E over X is real-analytically isomorphic (away from any singular loci) to the moduli space of solutions to the $t \neq 0$ Kapustin–Witten equations (4.3), (4.4) on E over X .*

This observation may motivate the study of the $P = W$ phenomenon developed for the nonabelian Hodge theory of curves by de Cataldo–Hausel–Migliorini [8] and others to the complex projective surface case. The $P = W$ conjecture posits that two different filtrations on the cohomologies of the moduli spaces coincide — the theorem above may connect the phenomenon to the topology of moduli spaces in the complex projective surface case and furthermore to their interpretation in terms of super Yang–Mills theory in physics. The relationship between opers and nonabelian Hodge-theoretic deformations, as per Dumitrescu–Fredrickson–Kydonakis–Mazzeo–Mulase–Neitzke [14], might also be discussed now in a higher-dimensional context.

5 The expected dimensions of the moduli spaces on closed four-manifolds

In the previous section, we saw the moduli spaces of solutions to the Kapustin–Witten equations (2.1)–(2.3) at different values of t are real-analytically isomorphic away from singular loci on a smooth, compact Kähler surface. A natural question to ask then is that to what extent of generality this might hold. In this section, we prove that the expected dimensions of the moduli spaces of solutions to the $t = 0$ Kapustin–Witten equations (2.4) and $t \neq 0$ Kapustin–Witten equations (2.5) coincide on a smooth, closed four-manifold.

The expected dimension is the dimension of the moduli space when it is smooth and unobstructed. This can be computed by considering Atiyah–Hitchin–Singer type deformation complex, which is formed from the system of linearised equations in each problem, and they are elliptic if the problem is well-posed. Then, the expected dimension is given by -1 times the index of the complex, since the Zariski tangent space of the moduli space is identified with the first cohomology of the complex.

To set the stage for this, let X be a closed, oriented, smooth four-manifold, and let $P \rightarrow X$ be a principal G -bundle over X , where G is a compact Lie group. We write $\Lambda_X^p := \Lambda^p T^*X$, for $p = 0, \dots, 4$. We denote by Λ_X^\pm the self-dual part and the anti-self-dual part of Λ_X^2 , respectively, and by $\pi^\pm: \Gamma(\text{ad}(P) \otimes \Lambda^2) \rightarrow \Gamma(\text{ad}(P) \otimes \Lambda^\pm)$ the projections. We define $d_A^\pm := \pi^\pm \circ d_A$.

5.1 The $t = 0$ case

For $t = 0$, the linearisation of the system of the equation (2.4) with gauge fixing fits into to the following Atiyah–Hitchin–Singer type complex:

$$\begin{aligned}
 0 &\longrightarrow \Gamma(\text{ad}(P) \otimes \Lambda_X^0) \xrightarrow{(d_A, [\mathfrak{a}, \cdot])} \Gamma(\text{ad}(P) \otimes (\Lambda_X^1 \oplus \Lambda_X^1)) \\
 &\xrightarrow{D_{A,t=0}} \Gamma(\text{ad}(P) \otimes \Lambda_X^+) \times \Gamma(\text{ad}(P) \otimes (\Lambda_X^- \oplus \Lambda_X^0)) \longrightarrow 0,
 \end{aligned}
 \tag{5.1}$$

where $D_{A,t=0}(a, \alpha) := (d_A^+ a + [\mathfrak{a} \wedge a]^+ + [\alpha \wedge \mathfrak{a}]^+, (d_A^- \alpha, d_A^* \alpha))$ for $(a, \alpha) \in \Gamma(\text{ad}(P) \otimes (\Lambda_X^1 \oplus \Lambda_X^1))$. This is a complex if (A, \mathfrak{a}) satisfies (2.4), and it is elliptic.

Recall from Definition 2.2 that $k(G) = c_2(P \otimes \mathbb{C})[X]$ if G is $SU(2)$ and $k(G) = -\frac{1}{4}p_1(P)[X]$ if G is $SO(3)$.

Theorem 5.1 *If the structure group G of P is one of $SU(2)$ or $SO(3)$, then the index of the elliptic complex (5.1) is given by $-16k(G) + 3\chi(X)$, where $\chi(X)$ is the Euler number of X , namely, $\chi(X) = \sum_{i=0}^4 (-1)^i b^i(X)$.*

Proof We ignore the zero-th order terms in (5.1), since the index only depends upon the symbol of the complex. Namely, we consider the following elliptic operator for the calculation of the index:

$$L_{A,t=0}: \Gamma(\text{ad}(P) \otimes (\Lambda_X^1 \oplus \Lambda_X^1)) \rightarrow \Gamma(\text{ad}(P) \otimes (\Lambda_X^+ \oplus \Lambda_X^- \oplus \Lambda_X^0 \oplus \Lambda_X^0)), \tag{5.2}$$

which is defined as $L_{A,t=0}(A, \alpha) := (d_A^+ a, d_A^- \alpha, d_A^* \alpha, d_A^* a)$ for

$$(A, \alpha) \in \Gamma(\text{ad}(P) \otimes (\Lambda_X^1 \oplus \Lambda_X^1)).$$

The operator (5.2) decomposes into the following two elliptic operators:

$$L^+: \Gamma(\text{ad}(P) \otimes \Lambda_X^1) \xrightarrow{(d_A^+, d_A^*)} \Gamma(\text{ad}(P) \otimes (\Lambda_X^+ \oplus \Lambda_X^0)), \tag{5.3}$$

$$L^-: \Gamma(\text{ad}(P) \otimes \Lambda_X^1) \xrightarrow{(d_A^-, d_A^*)} \Gamma(\text{ad}(P) \otimes (\Lambda_X^- \oplus \Lambda_X^0)), \tag{5.4}$$

These are familiar operators in gauge theory, and the indices are given by $8k(G) - 3(1 - b^1 + b^+)$ and $8k(G) - 3(1 - b^1 + b^-)$ respectively (see e.g. Atiyah–Hitchin–Singer [1, Section 6], Donaldson–Kronheimer [13, Section 4.2]), where b^+ and b^- is the dimension of maximal positive and negative definite subspaces of the intersection form on $H_2(X, \mathbb{Z})$ respectively. Since the elliptic operator (5.2) is the direct sum of the elliptic operators (5.3) and (5.4), the index of (5.2) is given by $16k(G) - 3(2 - 2b^1 + b^2)$. From the Poincaré duality, this equals to $16k(G) - 3(1 - b^1 + b^2 - b^3 + b^4)$, thus the assertion holds. \square

5.2 The $t \neq 0$ case

For $t \in \mathbb{R} \setminus \{0\}$, the linearisation of the system of the equation (2.5) with gauge fixing fits into to the following Atiyah–Hitchin–Singer type complex:

$$0 \longrightarrow \Gamma(\mathrm{ad}(P) \otimes \Lambda_X^0) \xrightarrow{(d_A, [\mathfrak{a}, \cdot])} \Gamma(\mathrm{ad}(P) \otimes (\Lambda_X^1 \oplus \Lambda_X^1)) \xrightarrow{D_{A,t} \neq 0} \Gamma(\mathrm{ad}(P) \otimes (\Lambda_X^2 \oplus \Lambda_X^0)) \longrightarrow 0, \quad (5.5)$$

where $D_{A,t} \neq 0(a, \alpha) := (d_A a + d_A \alpha + [\mathfrak{a} \wedge \alpha] + [\alpha \wedge \mathfrak{a}], d_A^* \alpha - *[\alpha \wedge * \mathfrak{a}])$ for $(a, \alpha) \in \Gamma(\mathrm{ad}(P) \otimes (\Lambda_X^1 \oplus \Lambda_X^1))$. This is a complex if (A, \mathfrak{a}) satisfies (2.5), and it is elliptic.

The following was obtained by Mazzeo–Witten [35, Proposition 4.1], they did it also for the case of four-manifolds with boundary. We give a proof of it here for the reader's convenience.

Theorem 5.2 (Mazzeo–Witten) *Consider a principal G -bundle over X , where G is $SU(2)$ or $SO(3)$. Suppose that $k(G) = 0$. Then the index of the elliptic complex (5.5) is given by $3\chi(X)$.*

Proof We ignore the zero-th order terms in (5.5) here as well. A trick here is to identify the second summand Λ_X^1 in the second term of (5.5) with Λ_X^3 and the second summand Λ_X^0 in the third term of (5.5) with Λ_X^4 by the Hodge star operator on X . Then by holding them into the odd degree part and the even degree part, the index calculation of (5.5) reduces to that of the following twisted de Rham operator:

$$L_{A,t} \neq 0: \Gamma(\mathrm{ad}(P) \otimes (\Lambda_X^1 \oplus \Lambda_X^3)) \rightarrow \Gamma(\mathrm{ad}(P) \otimes (\Lambda_X^0 \oplus \Lambda_X^2 \oplus \Lambda_X^4)). \quad (5.6)$$

This operator is defined by $L_{A,t} \neq 0(a, \alpha') := (d_A^* a, d_A a + d_A^* \alpha', d_A \alpha')$ for $(a, \alpha') \in \Gamma(\mathrm{ad}(P) \otimes (\Lambda_X^1 \oplus \Lambda_X^3))$. The topological index of the elliptic operator (5.6) is given by

$$-\frac{\mathrm{ch}(\mathrm{ad}(P) \otimes \mathbb{C}) \mathrm{ch}(\sum_{i=0}^4 (-1)^i \Lambda_{TX \otimes \mathbb{C}}^i) \mathrm{td}(TX \otimes \mathbb{C})}{e(TX)} [X],$$

where $\mathrm{ad}(P) \otimes \mathbb{C} := P \times_{\mathrm{ad}} (\mathfrak{g} \otimes \mathbb{C})$ (see e.g. Shanahan [42, I] for more details). From the assumption, we have $\mathrm{ch}(\mathrm{ad}(P) \otimes \mathbb{C}) = \dim \mathfrak{g} = 3$. In addition, the remaining factor

$$-\frac{\mathrm{ch}(\sum_{i=0}^4 (-1)^i \Lambda_{TX \otimes \mathbb{C}}^i) \mathrm{td}(TX \otimes \mathbb{C})}{e(TX)} [X]$$

is -1 times the topological index for the de Rham complex on X (see [42, II.4]), hence it equals -1 times the Euler number of X . Thus the assertion holds. \square

We immediately obtain the following:

Corollary 5.3 *Consider a principal G -bundle over a closed, oriented, smooth four-manifold X , where G is one of $SU(2)$ or $SO(3)$. Assume that $k(G) = 0$. Then the indices of (5.1) and (5.5) coincide. In other words, so do the expected dimensions of the moduli spaces of solutions to the Kapustin–Witten equations for $t = 0$ and $t \neq 0$ over X .*

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