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The Gromov limit for vortex moduli spaces

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We generalize the descriptions of vortex moduli spaces in [4] to more than one section with adiabatic constant s. The moduli space is topologically independent of s but is not compact with respect to C^∞ topology. Following [17], we construct a Gromov limit for vortices of fixed energy, and attempt to compactify the moduli space via bubble trees with possibly conical bubbles (or *raindrops*).

Keywords: Vortex equations; moduli spaces; bubbling; bubble tree.

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1. Introduction

The study of vortex equations finds its origin in Ginzburg–Landau's descriptions of the field configurations of superconducting material (cf. [11]). There, the energy functional is given in the form of Yang–Mills–Higgs functional, which depends on the electromagnetic potential D and the wave function ϕ of Cooper pairs of electrons. Stable configurations are governed by minimizing the energy functional, and the minimizing equations are known as the vortex equations.

The theory is mathematically modeled by equations on Hermitian vector bundles (E, H) of degree r over closed Kähler manifolds M, where the variables consist of $D \in \mathcal{A}(H)$, an H-unitary connection, and global smooth section ϕ. The Kähler form ω of M is normalized so that $Vol_\omega(M) = 1$. The Yang–Mills–Higgs energy functional contains two extra terms from classical Yang–Mills functional, arisen from sections:

$$YMH_{1,1}(D, \phi) := \|F_D\|^2_{L^2} + \|D\phi\|^2_{L^2} + \frac{1}{4}\|\phi \otimes \phi^{*H} - \tau\|^2_{L^2}, \tag{1.1}$$

where F_D is the curvature form of connection D and τ is a real parameter. The minimizing equations can be deduced by Bogomol'yi arguments ([4]):

$$\begin{cases} F_D^{(0,2)} = 0 \\ D^{(0,1)}\phi = 0 \\ \sqrt{-1}\Lambda F_D + \frac{1}{2}(\phi \otimes \phi^{*_H} - \tau) = 0. \end{cases} \tag{1.2}$$

Namely, among pairs (D, ϕ) such that D is integrable and ϕ is D-holomorphic, the last equation imposes a relation on mean curvature and the norm of the sections. As a standard principle in gauge theoretic equations, the existence of solutions is equivalent to a ϕ and τ dependent stability of the line bundle E (cf. [4, 5]). For line bundles $E = L$, the absence of proper subsheaves turns the stability condition dependent only on the parameter τ. In [4, 5], it is proved that the necessary condition for vortex to exist, followed by integrating the third equation of (1.2),

$$\tau \geq 4\pi r \tag{1.3}$$

is also sufficient. Solutions to (1.2) are clearly invariant under standard unitary gauge actions. Within the stable range, the moduli space of solutions to (1.2), or the gauge classes of *vortices*, has been explicitly described in [4]. The moduli space of vortices to (1.2) is precisely $Div_+^r M$, the space of degree r effective divisors on M, and is topologically independent of τ.

Generalizations of the classical results for the case of line bundles have been made in [2, 14]. We consider Yang–Mills–Higgs functional defined on $k+1$ sections with a scaling parameter s. The parameter τ can be absorbed into s (cf. [14]) and we rewrite

$$YMH_{k+1,s}(D, \phi) := \frac{1}{s^2}\|F_D\|_{L^2}^2 + \sum_{i=0}^{k} \|D\phi_i\|_{L^2}^2 + \frac{s^2}{4}\left\| \sum_{i=0}^{k} |\phi_i|_H^2 - 1 \right\|_{L^2}^2, \tag{1.4}$$

where ϕ is an abbreviation for $(\phi_i)_{i=0}^k$. The corresponding vortex equations are

$$\begin{cases} F_D^{(0,2)} = 0 \\ D^{(0,1)}\phi_i = 0 \quad \forall\, i \\ \sqrt{-1}\Lambda F_D + \frac{s^2}{2}\left(\sum_{i=0}^{k} |\phi_i|_H^2 - 1\right) = 0. \end{cases} \tag{1.5}$$

The stability condition is then

$$s^2 \geq 4\pi r. \tag{1.6}$$

Solutions to (1.5) are again invariant under unitary gauge group $\mathcal{G}$. We then define, for s in the stable range, the gauge class of solutions:

Definition 1.1.

$$\nu_{k+1}(s) := \{(D, \phi) \in \mathcal{A}(H) \times \Omega^0(L) \times \cdots \times \Omega^0(L) \mid (1.5)\text{hold}\}/\mathcal{G}.$$

These spaces are topologically independent of s and fibers over a space with explicit description. Detailed descriptions are provided in Sec. 3. Each class $[D_s, \phi_s] \in \nu_{k+1}(s)$ represents a unique holomorphic structure for the line bundle L. Nevertheless, the topological structures of the line bundle L, determined by the mean value of the corresponding curvatures F_{D_s}, remain undisturbed until the adiabatic limit $s = \infty$. As noted in [2], the formal limit of (1.5) as $s \to \infty$ is:

$$\begin{cases} F_D^{(0,2)} = 0 \\ D^{(0,1)}\phi = 0 \\ \displaystyle\sum_{i=0}^{k} |\phi_i|_H^2 - 1 = 0, \end{cases} \tag{1.7}$$

signaling some topological distinctions from that of (1.5). In particular, for the case of one section $k = 0$, the third equation above requires the global section to be non-vanishing, which only exists on trivial line bundles. Moreover, the norms of the sections are no longer constrained by curvature. These inconsistencies signal bubbling phenomenon of vortex moduli spaces along some variation of vortices. The main theme of this paper is to study such situations and provide explicit description of the bubble formation. Moreover, we construct a reasonable limiting object for $[D_s, \phi_s]$ without topological loss.

The bubbling phenomenon depends crucially on the dynamics of solutions to (1.5) and occurs when singularities form in the limiting solution. As will be shown in Sec. 5, singularities of limiting solutions are due to accumulations of common zeros, or base points, of ϕ_i's at the boundary. In particular, consider the generic open subset

Definition 1.2.

$$\nu_{k+1,0}(s) := \{[D, \phi_0, \ldots, \phi_k] \in \nu_{k+1}(s) \mid \cap_i \, \phi_i^{-1}(0) = \emptyset\}.$$

Analytic results from [14] show that no bubbling phenomenon occurs when the entire convergence takes place in $\nu_{k+1,0}(s)$. We are therefore mainly interested in families of vortices for which new common zeros form at infinity. That is, when the sequence in $\nu_{k+1,0}(s)$ converges to a boundary point. The dense open subset above may be identified diffeomorphically by a family of holomorphic maps from M to $\mathbb{CP}^k$ (cf. [8]). The Dirichlet energies of these associated maps are precisely the degree of the bundle. For closed Riemann surfaces $M = \Sigma$, these are holomorphic curves with bounded energies and we may then apply results from [9, 17] to establish their convergence behaviors of Gromov type. The limiting behaviors and objects known as the "bubble trees" are compatible with vortex moduli spaces. The descriptions of bubble trees require some amount of work. For a fixed line bundle L over a closed Riemann surfaces Σ, we first prove the following theorem.

Theorem 1.3 (Theorem 5.1 on *Formal Removal of Singularities*). *Fix a Hermitian line bundle (L, H) over Σ. Given a sequence of vortices $\{[D_s, \phi_s]\} \subset \nu_{k+1,0}(s)$ approaching the boundary of $\nu_{k+1}(s)$, there exists a finite set of points $\{p_1, \ldots, p_N\} \subset \Sigma$, integers $\{a_1, \ldots, a_N\} \subset \mathbb{N}$ such that $\sum_j a_j \leq r$, and vortices $[D'_s, \phi'_s]$ with smooth (subsequential) limit $[D_0, \phi_0]$ on line bundle*

$$L_0 := L \otimes_j \mathcal{O}(-a_j p_j),$$

such that

- $[D'_s, \phi'_s] = [D_s, \phi_s]$ *on* $\Sigma \backslash \{p_1, \ldots, p_N\}$ *(via the isomorphism* $L_0 \simeq L$ *on* $\Sigma \backslash \{p_1, \ldots, p_N\}$*).*
- D'_s *and* ϕ'_s *satisfy the vortex equation*

$$\begin{cases} D_s'^{(0,1)} \phi'_{s,i} = 0 \quad \forall\, i \\ \sqrt{-1} \Lambda F_{D'_s} + \dfrac{s^2}{2} \left(\displaystyle\sum_{i=0}^{k} |\phi'_{s,i}|_H^2 - 1 \right) = 0, \end{cases} \tag{1.8}$$

  on $L_0 \to \Sigma$.
- $[D_0, \phi_0]$ *satisfies*

$$\begin{cases} D_0^{(0,1)} \phi_{0,i} = 0 \quad \forall\, i \\ \displaystyle\sum_{i=0}^{k} |\phi_{0,i}|_H^2 - 1 = 0, \end{cases} \tag{1.9}$$

  on $L_0 \to \Sigma$.

This theorem basically states that a sequence of vortices on a line bundle approach one with singularities and we may formally remove the singularities. Details will be fully explained in Sec. 5.

The extended line bundle L_0 is of degree $r - \sum_j a_j$. The reduction of degree suggests concentration of energy of vortices near singularities. Appropriately rescaling nearby coordinates by some factor $t_j(s)$, we may smooth out the energy density and define vortices on $\mathbb{C}$, which is identified by $\mathbb{S}^2 \backslash \{p^+\}$, where p^+ is the north pole, via stereographic projection. The limiting objects are determined by the rates of energy blow ups.

Theorem 1.4 (Theorem 5.2 on *Renormalization*). *For each p_j in Theorem 1.3, there exists $\epsilon > 0$ so that the geodesic disc $B(p_j, \epsilon)$ is conformally equivalent to $B_s \subset \mathbb{S}^2$, an increasing family of domains with $\cup_s B_s = \mathbb{S}^2 \backslash \{p^+\}$, and the followings hold*:

- *The pullbacked vortices* $[D_s^*, \phi_s^*]$ *on* B_s*, satisfying*

$$\begin{cases} D_s^{*0,1} \phi_{s,i}^* = 0 \\ \sqrt{-1} \Lambda_s^* F_{D_s^*} + \dfrac{s^2}{2 t_j(s)^2} \left(\displaystyle\sum_{i=0}^{k} |\phi_{s,i}^*|_H^2 - 1 \right) = 0 \end{cases} \tag{1.10}$$

on pullbacked line bundle L_s *over* B_s*, coincide with the vortices defined by pullbacked holomorphic functions* $\tilde{f}_s : B_s \to \mathbb{CP}^k$*.*

- *Exactly one of the followings holds true*:
  (a) *There exists a* C^1_{loc}*-convergent subsequence of* $\{[D^*_s, \phi^*_s]\}$ *whose limit* $[D_j, \phi_j]$ *satisfies*

$$\begin{cases} D_j \phi_{j,i} = 0 \\ \displaystyle\sum_{i=0}^{k} |\phi_{j,i}|^2_H - 1 = 0 \end{cases} \tag{1.11}$$

  defined on the entire $\mathbb{S}^2$*. That is, a holomorphic sphere in* $\mathbb{CP}^k$ *bubbles off.*
  (b) *There exsits points* $\{p^1_j, \ldots, p^{N_j}_j\} \subset \mathbb{S}^2$*, non-negative integers* $a^0_j, a^1_j, \ldots, a^{N_j}_j$*, and a* C^1_{loc}*-convergent subsequence of* $\{[D^*_s, \phi^*_s]\}$ *on* $\mathbb{S}^2 \backslash \{p^1_j, \ldots, p^{N_j}_j, p^+\}$*, whose limit* $[D_j, \phi_j]$ *satisfies*

$$\begin{cases} D_j \phi_{j,i} = 0 \\ \sqrt{-1}\Lambda^* F_{D_j} + \dfrac{1}{2}\left(\displaystyle\sum_{i=0}^{k} |\phi_{j,i}|^2_H - 1\right) = 0 \end{cases} \tag{1.12}$$

  on a degree a^0_j *line bundle* L_j *over* $\mathbb{S}^2 \backslash \{p^1_j, \ldots, p^{N_j}_j, p^+\}$*. Moreover,* (L_j, D_j, ϕ_j) *is the* C^1 *limit of* (L^*_s, D^*_s, ϕ^*_s)*.*
- *On* $\mathbb{S}^2$*,* $\sqrt{-1}\Lambda^* F_{D_j}$ *is a distribution given by a smooth function plus* $\sum_{l=1}^{N_j} a^l_j \delta(p^l_j)$*.*

Once again, full explanation will be given in Sec. 5. The principle to obtain the results above is to associate each generic vortex with a holomorphic map from Σ to $\mathbb{CP}^k$ (see Sec. 3) and apply analytic results from [17]. The limiting vortex $[D_j, \phi_j]$ above is given by a limiting map $\tilde{f}_{p_j}$ as well. Standard Morrey estimate and bootstrapping arguments allow one to extend $\tilde{f}_{p_j}$ holomorphically to the entire $\mathbb{S}^2$. Analogous extension is possible for vortices but requires certain adjustments.

Theorem 1.5 (Removal of Singularities for Limiting Vortices). *Continuing with the setting of Theorems* 1.3 *and* 1.4, *the conformal transformations may be modified so that the limiting vortex* $[D_j, \phi_j]$ *may be extended, after an appropriate gauge transformation, across* p^+*. The extended pair gives rise to a vortex defined on a non-trivial line bundle* L_j *over* $\mathbb{S}^2$ *with degree* $\leq a_j$*. The metric on* $\mathbb{S}^2$ *may come with a conic singularity.*

To this end we have associated a "bubble", meant to smoothen the energy spike, to each point where energy density blows up (the "bubble point"). The description is not yet satisfactory because the inequality at the end of the theorem above may be strict. That is, the bubble does not necessarily retain all the energy near p_j. Following [17], we may modify the renormalization process to accurately account for all the energy concentrated near each p_j by a finite sequence of bubbles. At the end, we obtain a bubble tree T, a wedge sum of Σ and $\mathbb{S}^2$'s. The vortices $[D_s, \phi_s]$

"Gromov converges" to a vortex $[\mathcal{D}, \varPhi]$ on T. The vague terms here will be precisely defined in the due course.

Theorem 1.6 (Theorem 6.10 on *Bubble Tree*). *The vortices $V_s := \{[D_s, \phi_s]\}$ on a degree r line bundle L over Σ Gromov converge to a vortex $V := [\mathcal{D}, \varPhi]$ over a degree r line bundle $\mathcal{L}$ over a bubble tree T defined by*

$$T := T_0 \vee T_1 \vee \cdots \vee T_{N_V}, \tag{1.13}$$

where $T_0 = \Sigma$. For $n \geq 1$, each T_n is a disjoint union of 2-spheres with either round metric $g_{\mathbb{S}^2}$ or conic metric g_β.

2. Background and Established Results

This section briefly summarizes results from [4, 14] on the geometric descriptions of some special $\nu_{k+1}(s)$'s. Analytic techniques developed in [4] assume $k, s = 1$, but they are by no means special to this particular values. We will therefore cite those results with general $k \in \mathbb{N}$ and s in the stable range. Readers familiar with these work may skip to the next section.

Let (M, ω) be a closed Kähler manifold of unit volume. (L, H) is a Hermitian line bundle of degree r over it. Denote by $\mathcal{A}(H)$ the space of H-unitary connections and $\Omega^0(L)$ the space of smooth global sections. The symmetries of (L, H) considered are denoted by $\mathcal{G}_{\mathbb{C}}$ and $\mathcal{G}$, called the complex gauge group and unitary gauge groups, respectively. Both groups act on $\mathcal{A}(H)$, $\Omega^0(L)$, the metrics, complex structures, and curvature forms of L in the standard ways (see [4] or [12]). The necessity of the stability condition $s^2 \geq 4\pi r$ implies immediately that $\nu_{k+1}(s) = \emptyset$ for all s with $s^2 < 4\pi r$. For the critical value $s^2 = 4\pi r$, the third equation of (1.5) requires that all sections ϕ_i to be trivial, and therefore $\nu_{k+1}(s)$ is precisely $\mathcal{A}(H)/\mathcal{G}_{\mathbb{C}}$, or the space of holomorphic structures of L. For $M = \Sigma$, the space corresponds to the Jacobian torus of degree r, $Jac^r\Sigma$.

Analytic discussion enters when $s^2 > 4\pi r$. Similar to the search of Hermitian–Einstein connections, solving the tensorial vortex equation modulo unitary gauge group is equivalent to searching for special Hermitian metric modulo complex gauge group. The restatement of the problem by variation of metrics invites classical analytic tools from [13] to enter the central argument.

We briefly summarize the correspondence of the two aspects. For a Hermitian line bundle (L, H), it is a classical fact that the space of unitary connection $\mathcal{A}(H)$ and the space $\mathcal{C}$ of holomorphic structures, or the collection of $\mathbb{C}$-linear operators

$$\bar{\partial}_L : \Omega^0(L) \to \Omega^{0,1}(L)$$

satisfying Leibiniz rule and $\bar{\partial}_L \circ \bar{\partial}_L = 0$, identify each other. Fix $k+1$ global sections $\phi = (\phi_i)_{i=0}^k$. The original tensorial approach to solve (1.5) is then equivalent to finding a holomorphic structure $\bar{\partial}_L$, that makes all ϕ_i holomorphic, so that the corresponding unitary connection D and curvature F_D satisfy the equation. This approach, however, is rather abstract. The alternative, or the scalar approach, picks an arbitrary pair in the space of *holomorphic pair*

$$\mathcal{N}_{k+1} := \{(\bar{\partial}_L, \phi) \in \mathcal{C} \times \Omega^0(L) \times \cdots \times \Omega^0(L) \,|\, \bar{\partial}_L(\phi_i) = 0 \;\forall\, i\}. \tag{2.1}$$

We then look for a special $H_s \in \mathcal{H}$, the space of Hermitian structure, whose corresponding connection, and therefore curvature form F_{D_s}, together with the given sections satisfy the third equations of (1.5):

$$\sqrt{-1}\Lambda F_{D_s} + \frac{s^2}{2}\left(\sum_{i=0}^{k} |\phi_i|^2_{H_s} - 1\right) = 0. \tag{2.2}$$

For a line bundle L, the complex gauge group $\mathcal{G}_{\mathbb{C}}$ acts transitively on $\mathcal{H}$. In particular, the special metric H_s and the background metric H are related by

$$H_s = e^{2u_s}H,$$

where u_s is a real smooth function on M. The corresponding curvature F_{D_s} is then related to the background curvature F_H by

$$\sqrt{-1}\Lambda F_{D_s} = \sqrt{-1}\Lambda F_H - \Delta u_s, \tag{2.3}$$

where Δ is the positive definite Laplacian determined by the Kähler form ω. Let $c_1 = 2\pi r$, and $c(s) = 2c_1 - \frac{s^2}{2}$, which is negative for s in the stable range. Also let ψ be the unique solution to the Poisson equation

$$\Delta\psi = \sqrt{-1}\Lambda F_H - c_1. \tag{2.4}$$

It can be readily verified that solving (2.2) above is equivalent to solving the following *Kazdan–Warner equation*

$$\Delta\varphi_s + \frac{s^2}{2}he^{\varphi_s} - c(s) = 0, \tag{2.5}$$

where $\varphi_s = 2(u_s - \psi)$ and the norm function

$$h = -e^{2\psi}\sum_{i=0}^{k} |\phi_i|^2_H \tag{2.6}$$

is non-positive and vanishes precisely at the common zeros of ϕ_i's, an effective divisor $\mathcal{E}$ of degree $\leq r$. For these choices of $c(s)$ and h, techniques developed in [13] guarantee a unique smooth solution φ_s for each finite s in the stable range. The unique existence is proved by the standard arguments of upper and lower solutions of elliptic operators and applications of maximum principles.

The analytic results imply that given a holomorphic pair $(\bar{\partial}_L, \phi) \in \mathcal{N}_{k+1}$ in (2.1), the special metric H_s to solve (2.2) is uniquely determined if $s^2 > 4\pi r$. One may readily recognizes the gauge ambiguities and conclude that the space of $\mathcal{G}$-classes of solutions to (1.5) corresponds bijectively to $\mathcal{N}_{k+1}/\mathcal{G}_{\mathbb{C}}$. For $k = 0$, this space is further identified, up to a $\mathcal{G}_{\mathbb{C}}$ action, with the space $Div^r_+(M)$ of effective divisor of degree r (cf. [4]). Indeed, an effective divisor of degree r determines a holomorphic line bundle $L = \mathcal{O}(\mathcal{E})$ with holomorphic structure $\bar{\partial}$, and all global sections vanishing along $\mathcal{E}$ are in one complex gauge orbit. We have,

Theorem 2.1 (Description of $\nu_1(s)$ for M).

$$\nu_1(s) = \begin{cases} \emptyset; & s^2 < \pi r \\ \mathcal{A}(H)/\mathcal{G}_{\mathbb{C}}; & s^2 = 4\pi r \\ Div_+^r(M); & s^2 > 4\pi r. \end{cases}$$

For a closed Riemann surface $M = \Sigma$, $Div_+^r(\Sigma)$ is precisely the space of unordered r points, or the symmetric space and $\mathcal{A}(H)/\mathcal{G}_{\mathbb{C}}$ is identified with the Jacobian torus of degree r. We have

Theorem 2.2 (Description of $\nu_1(s)$ for Σ).

$$\nu_1(s) = \begin{cases} \emptyset; & s^2 < \pi r \\ Jac^r\Sigma; & s^2 = 4\pi r \\ Sym^r\Sigma; & s^2 > 4\pi r, \end{cases}$$

$\nu_{k+1}(s)$ for general k has been described in [6] for the case $M = \Sigma$. In the next section, we provide a general description which can be easily specialized to the case of Riemann surfaces. Since we are interested in the adiabatic limit $s \to \infty$, we will from now on assume $s^2 > 4\pi r$, ruling out the first two possibilities in Theorems 2.1 and 2.2.

3. Generalized Vortex Moduli Spaces and Maps to Projective Spaces

We provide the general descriptions for $\nu_{k+1}(s)$ here. In the space $\mathcal{N}_1$ in (2.1), we see that after a complex structure $\bar{\partial}_L$ is fixed, holomorphic sections are determined by effective divisors of degree r up to $\mathcal{G}_{\mathbb{C}}$ gauge. For holomorphic tuples with $k+1$ sections, it is natural to analogously identify each vortex by the tuple of $k+1$ divisors defined by each section. However, ambiguities and restrictions arise. An immediate restriction is that all divisors must define isomorphic holomorphic line bundle, as we are fixing one holomorphic structure at a time. In another words, all divisors must be linearly equivalent. Therefore we start with the space

Definition 3.1.

$$\mathbb{E}_{k+1} := \{(E_0, \ldots, E_k) \in (Div_r^+(M))^{\times k+1} \mid E_0 \sim \cdots \sim E_k\}.$$

This is a closed subset of $(Div_r^+(M))^{\times k+1}$ and therefore compact. It however is not an identification of $\nu_{k+1}(s)$. An effective divisor E_i determines a global section ϕ_i up to an element in $\mathcal{G}_{\mathbb{C}}$, and the gauge ambiguity for each i need not be unitary gauge equivalent. That is, $\mathbb{E}_{k+1}$ only identifies $\mathcal{N}_{k+1}$ up to a $(\mathcal{G}_{\mathbb{C}})^{k+1}$ action, which is larger than the diagonal action $\mathcal{G}_{\mathbb{C}}$ on $\mathcal{N}_{k+1}$ used to define $\nu_{k+1}(s)$. Therefore, each element in $\mathbb{E}_{k+1}$ determines a vortex element in $\nu_{k+1}(s)$ up to a $(\mathcal{G}_{\mathbb{C}})^{k+1}/\mathcal{G}_{\mathbb{C}}$ orbit. More precisely, we have

Theorem 3.2. *The space $\nu_{k+1}(s)$ fibers over the space $\mathbb{E}_{k+1}$ with toric fiber $(\mathbb{C}^*)^{k+1}/\mathbb{C}^*$.*

Proof. For each $p \in M$, let $U_p \subset \mathbb{E}_{k+1}$ be the open subset such that no divisor contains p. Since $\mathbb{E}_{k+1}$ is compact, it may be covered by finitely many such open subsets $U_1, \ldots, U_M$, each equipped with a based point β_j away from all divisors in U_j. Each $(E_0, \ldots, E_k) \in U_j$ determines a holomorphic structure $\bar{\partial}_L$ of L (i.e. $L \simeq \mathcal{O}(E_0) \simeq \cdots \simeq \mathcal{O}(E_k)$). Every E_i determines a global holomorphic section ϕ_i up to a non-zero constant. Therefore, two such sections are equal if and only if their values are equal at the based point β_j. Explicitly, consider the projection

$$\tilde{\pi} : \mathcal{N}_{k+1} \to \mathbb{E}_{k+1}$$

defined by

$$\tilde{\pi}(\bar{\partial}_L, \phi_0, \ldots, \phi_k) = (\phi_0^{-1}(0), \ldots, \phi_k^{-1}(0)).$$

This is a fiber bundle with fiber $(\mathbb{C}^*)^{k+1}$, where the local trivialization over U_j is given by

$$\tilde{\rho}_j(\bar{\partial}_L, \phi_0, \ldots, \phi_k) = (\phi_0^{-1}(0), \ldots, \phi_k^{-1}(0), \phi_0(\beta_j), \ldots, \phi_k(\beta_j)).$$

Finally we note that in the identification $\nu_{k+1}(s) \simeq \mathcal{N}_{k+1}/\mathcal{G}_{\mathbb{C}}$, the gauge action does not affect the zeros of the sections and therefore the projection $\tilde{\pi} : \mathcal{N}_{k+1} \to \mathbb{E}_{k+1}$ descends to the quotient $\pi : \nu_{k+1}(s) \to \mathbb{E}_{k+1}$. It follows that $\nu_{k+1}(s)$ is a bundle over $\mathbb{E}_{k+1}$ with fiber $(\mathbb{C}^*)^{k+1}/\mathbb{C}^*$. □

The theorem is clearly consistent with [4], where $k = 0$.

To relate $\nu_{k+1}(s)$ to the space of holomorphic maps, we consider its open dense subset:

Definition 3.3.

$$\nu_{k+1,0}(s) := \{[D, \phi_0, \ldots, \phi_k] \in \nu_{k+1}(s) \mid \cap_i \phi_i^{-1}(0) = \emptyset\}.$$

On this subset, we may control the special gauges (i.e. solutions to (2.5)) and guarantee the existence of their smooth limit. This is due to the fact that the norm function h in (2.6) is strictly negative, and we may uniformly bound super and subsolutions for the elliptic Kazdan–Warner equations (2.5). The general analytic statement from [14] is

Theorem 3.4 (Asymptotic Behaviors on $\nu_{k+1,0}(s)$). *On a compact Riemannian manifold M without boundary, let c_1 be any constant, c_2 any positive constant, and h any negative smooth function. Let $c(s) = c_1 - c_2 s^2$, for each s large enough, the unique solutions $\varphi_s \in C^\infty(M)$ for the equations*

$$\Delta\varphi_s = c(s) - s^2 h e^{\varphi_s}$$

are uniformly bounded in $H^{l,p}$ *for all* $l \in \mathbb{N}$ *and* $p \in [1, \infty]$. *Moreover, in the limit* $s \to \infty$, φ_s *converges smoothly* (*i.e. uniformly in all* $H^{l,p}$) *to*

$$\varphi_\infty = \log\left(\frac{c_2}{-h}\right),$$

the unique solution to

$$he^{\varphi_\infty} + c_2 = 0.$$

The theorem in particular rules out the formation of bubble point away from the boundary of $\nu_{k+1}(s)$.

Topologically, the space $\nu_{k+1,0}(s)$ is identified with $\mathcal{H}_{r,k}$, the space of degree r holomorphic maps from M to $\mathbb{CP}^k$. For every $f \in \mathcal{H}_{r,k}$, consider the following background data: On the anti-tautological line bundle $\mathcal{O}(1)$ over $\mathbb{CP}^k$ and its pullbacked bundle $L = f^*\mathcal{O}(1)$ over M, let $\phi := (\phi_i)_{i=0}^k$ be the global holomorphic (with respect to the pulled back holomorphic structure) sections on L pullbacked from the hyperplane sections $z_0, \ldots, z_k$ on $\mathcal{O}(1)$ via f. $\mathcal{O}(1)$ is equipped naturally with the Fubini–Study metric, which is also pulled back to be the background metric H on L, and therefore determines an unitary connection D. We then gauge transform the initial data $[D, \phi]$ with complex gauge determined by Kazdan–Warner equation (2.5) into $[D_s, \phi_s]$ that solves vortex equations. These sections clearly have no common zeros, and we define

$$\Phi_s : \mathcal{H}_{r,k} \to \nu_{k+1,0}(s) \tag{3.1}$$

by

$$\Phi_s(f) = [D_s, \phi_s].$$

The correspondence is in fact a diffeomorphism with a natural inverse

$$\Phi_s^{-1} : \nu_{k+1,0}(s) \to \mathcal{H}_{r,k}$$

given by

$$\Phi_s^{-1}([D_s, \phi_s])(p) := f_s(p) = [\phi_{s,0}(p) : \cdots : \phi_{s,k}(p)].$$

As a smooth map, f_s is clearly well defined as sections do not vanish simultaneously. Also, a different choice of trivilization amounts to multiplication of all components with a non-zero constant and therefore does not alter the definition. Moreover, each f_s is holomorphic with respect to the complex structure of Σ given by D_s, which corresponds to conformal deformations of Kähler form ω. Therefore the complex structure is independent of s and the map Φ_s^{-1} is indeed well defined. See for example, [14] for detailed verifications.

In the context of Theorem 3.2, $\nu_{k+1,0}(s)$ is the restriction of $\nu_{k+1}(s)$ over the generic open subset

$$\mathbb{E}_{k+1,0} := \{(E_0, \ldots, E_k) \in \mathbb{E}_{k+1} | \cap_i E_i = \emptyset\}.$$

For Riemann surface $M = \Sigma$, we have an explicit interpretation of Theorem 3.2 in terms of holomorphic maps, introduced in [6]. Indeed, for $M = \Sigma$, $Div_r^+(\Sigma) = Sym^r\Sigma$, the symmetric r product. For an element $(E_0, \ldots, E_k) \in \mathbb{E}_{k+1}$, we consider the divisor formed by their intersection and a holomorphic map from Σ to $\mathbb{CP}^k$ by the Theorem of Abel–Jacobi. Precisely, let $E = \cap_i E_i \in Sym^l\Sigma$, counting multiplicities, for $0 \leq l \leq r$ and $E_i' = E_i - E$. For each $i \geq 1$, the divisor $E_i' - E_0'$ has value zero under Abel–Jacobi map and therefore determines a meromorphic function ϕ_i on Σ. These maps have no common zero and define a holomorphic map of degree $r - l$ locally given by

$$f_\phi(z) := [1, \phi_1(z), \ldots, \phi_k(z)].$$

The map is unique up to a choice of image of the base point, which corresponds to a choice of representative from the fiber $(\mathbb{C}^*)^{k+1}/\mathbb{C}^*$ in Theorem 3.2. The representative is determined by Kazdan–Warner equation. We therefore uniquely associate a vortex $[D, \phi] \in \nu_{k+1}(s)$ with an element $(E, f_\phi) \in Sym^l \times \mathcal{H}_{r-l,k}$. This is the well known identification of $\nu_{k+1}(s)$ with the *Uhlenbeck Compactification* from [6].

Theorem 3.5 (Uhlenbeck Compactification [6]). *For all finite s large enough, the space $\nu_{k+1}(s)$ is homeomorphic to the stratification*

$$\overline{\mathcal{H}}_{r,k} := \bigsqcup_{l=0}^{r} (Sym^l\Sigma \times \mathcal{H}_{r-l,k}). \tag{3.2}$$

The topology of Uhlenbeck compactification is given sequentially. $(E_i, f_i) \to (E, f)$ if an only if

- $f_i \to f$ in $\mathcal{C}_0^\infty(\Sigma - E)$ topology, and
- $e(f_i) \to e(f)$ in weak* topology.

Here, $e(f)$ is the energy density $|df|^2$ of f with respect to ω and Fubini–Study metric on $\mathbb{CP}^k$. The weak convergence above says that for all $g \in \mathcal{C}^\infty(\Sigma)$, we have

$$\int_\Sigma ge(f_i) := \int_\Sigma gf_i^*\omega_{FS} \to \int_\Sigma gf^*\omega_{FS} := \int_\Sigma ge(f)$$

as $i \to \infty$. The precise homeomorphic correspondence is exhibited therein.

4. Gromov Compactness

Having identified $\nu_{k+1,0}(s)$ with $\mathcal{H}_{r,k}$, we intend to study convergence behaviors of vortices in terms of maps. We state relevant definitions and theorems from [17] regarding Gromov compactness of the space of jJ-holomorphic maps in the form applicable to the aimed results. From this section on, we discuss the case $(M, \omega) = (\Sigma, \omega)$, a closed Kähler Riemann surface with complex (conformal) structure j compatible with ω. Also, fix a compact symplectic manifold (Z, λ) with almost complex structure J. For our applications, we may assume that J is compatible with λ, and therefore λ-tamed. When no confusion arises, we use the same notation for energy density function

$$f^*\lambda := e(f)\omega_\Sigma$$

and the measure it represents on $C^\infty(\Sigma)$. Proofs from [17] will be reproduced only when they are relevant to our applications.

We need an important estimate on the energy density function by its integral, a standard result following from Bochner type estimate on $\Delta e(f)$. The inequality allows one to apply the Theorem of Arzela–Ascoli.

Proposition 4.1 ([17, Theorem 2.3] on *Energy Estimate*). *There exists positive constants C and ϵ_0, depending only on the complex geometry of Σ, such for all C^1 J-holomorphic map $f : \Sigma \to Z$ and geodesic disc $B(2r)$ with radius $2r$, with the property that*

$$E(2r) := \int_{B(2r)} e(f) \le \epsilon_0, \tag{4.1}$$

we have

$$\sup_{B(r)} e(f) \le \frac{C}{r^2} E(2r). \tag{4.2}$$

The removability of singularity of vortices depends on the following basic theorem on extension jJ-holomorphic map across a punctured disc.

Theorem 4.2 ([17, Theorem 3.7] on *Removable Singularity for Maps*). *Let (Z, J) be a complex manifold with complex structure J and (Σ, j) be a Riemann surface with complex structure j. Let $B\backslash\{p\}$ be a punctured disc of Σ and*

$$f : B\backslash\{p\} \to Z$$

be a jJ-holomorphic map of finite energy. Then f extends to a jJ-holomorphic map

$$\bar{f} : B \to Z.$$

The extension and its regularity follow from Morrey's Lemma ([16, Theorem 2.1]), and the required energy estimate is achieved by the strong isoperimetric inequality cited below. We will also need the fact that the energies of non-constant holomorphic maps can not be arbitrarily small.

Theorem 4.3 ([17, Proposition 1.1(b)]). *There is a constant $B_0 > 0$ such that any jJ-holomorphic map $f : \Sigma \to Z$ with energy less than B_0 is a constant map.*

The constant B_0 only depends on the complex structure and metric of Σ.

Next we cite two estimates that are needed to control energy loss. The first result imposes a lower bound for the area of the image of holomorphic maps.

Proposition 4.4 ([17, Corollary 3.2] on *Monotonicity*). *Let $f : \Omega \to Z$ be a jJ-holomorphic map on a domain $\Omega \subset \Sigma$. There is a constant c such that for any* *sufficiently small ball* $B(p,\delta)$ *in* Z *with center* $p \in f(\Omega)$ *and no boundary component inside* $B(p,\delta)$ *such that*

$$\mathrm{Area}(f(\Omega) \cap B(p,\delta)) \geq c\delta^2.$$

The second statement is a strong isoperimetric inequality for holomorphic maps.

Proposition 4.5 ([17, Proposition 3.4] on *Strong Isoperimetric Inequality*). *Let* $f : \Omega \to Z$ *be a* jJ*-holomorphic map on a domain* $\Omega \subset \Sigma$ *with boundary* $\partial\Omega$. *There are constants* ϵ_0, C *such that for any tamed* jJ*-holomorphic map* $f : \Omega \to Z$ *with length* $(f(\partial\Omega)) \leq \epsilon_0$, *there is an associated homology class* $\alpha_f \in H_2(Z,\mathbb{Z})$ *such that*

$$\mathrm{Area}(f(\Omega)) \leq C[\langle\lambda, \alpha_f\rangle + \mathrm{length}^2(f(\partial\Omega))]. \tag{4.3}$$

The bracket is the usual pairing and the quantity is known as the "symplectic area" of α_f.

Note that the two estimates above hold for any metric since any two metrics are uniformly equivalent due to the compactness of Z and tamedness of f.

The expository bubbling result is then the following theorem:

Theorem 4.6 ([17, Theorem 4.1] on *Bubbling of Holomorphic Curves*). *Given a sequence* $\{f_s\}$ *of* jJ*-holomorphic maps* $\Sigma \to Z$ *with uniformly bounded energies*:

$$E(f_s) := \int_\Sigma e(f_s) < C,$$

there is a subsequence still denoted by $\{f_s\}$, *a finite set of points* $\{p_1, \ldots, p_N\} \subset \Sigma$, *and a* jJ*-holomorphic map* $f_0 : \Sigma \to Z$ *such that*

(a) $f_s \to f_0$ *in* C^1 *on* $\Sigma\backslash\{p_1, \ldots, p_N\}$.
(b) *The energy densities* $e(f_s)$ *converge as measures to* $e(f_0)$ *plus a sum of Dirac-delta measures*:

$$e(f_s) \to e(f_0) + \sum_{j=1}^{N} a_j \delta(p_j) \tag{4.4}$$

where $a_j \geq B_0\ \forall j$. B_0 *is the constant in Theorem* 4.3.

The proof of this theorem contains critical renormalization techniques which induce holomorphic maps from $\mathbb{S}^2$ to Z, or "bubbles". We reproduce the proof below in our context.

Proof. The subsequence of $\{f_s\}$ exists due to compactness of Σ, the uniform boundedness of energies, and the Theorem of Arzela–Ascoli. For convenience, we do not change indices when extracting subsequences.

Take the constant $\epsilon_0 > 0$ as in Proposition 4.1. For each $m \in \mathbb{N}$, it is possible to cover Σ with a finite number of discs with radius $r_m = 2^{-m}\epsilon_0$. Denote these discs by $\{B(y_\alpha, r_m)\}$. The compactness of Σ allows us to further assume that $\{B(y_\alpha, \frac{r_m}{2})\}$ continues to cover Σ and that $r_m < inj_g$, the *injectivity radius* of the chosen background metric. Furthermore, we may assume that each point of Σ is covered by these discs at most M times, where M is uniform for all s and m. With these choices, and the energy bound, we may conclude that for each s,

$$\int_{B(y_\alpha, r_m)} e(f_s) < \epsilon_0$$

for all but finitely many m. By passing to a subsequence we may fix these discs for each m and conclude that

$$\int_{B(y_\alpha, r_m)} e(f_s) \geq \epsilon_0$$

only at certain "bad discs" $\{B(p_{1,m}, r_m), \ldots, B(p_{N,m}, r_m)\}$, where l is uniform over s. For each m, the estimate (4.2) holds on all "good discs", namely, those $B(y_\alpha, \frac{r_m}{2})$'s disjoint from bad discs. Since $e(f_s)$ dominates the first derivatives of f_s, we see that f_s and their first derivatives are uniformly bounded. By the Theorem of Arzela–Ascoli, we have a subsequence of $\{f_s\}$ that converge in C^1 on each good discs. On the other hand, as $m \to \infty$, the centers of bad discs converge (by a subsequence) to $\{p_1, \ldots, p_N\}$. These points are known as the *bubble points*. Picking the diagonal subsequence from the double sequence in s and m, we conclude the existence of $f_0 : \Sigma \backslash \{p_1, \ldots, p_N\} \to Z$ in part (a). The domain of f_0 can be extended holomorphically to Σ by Theorem 4.2. This establishes part (a).

Part (b) concerns energy densities on the bad discs $\{B(p_j, \epsilon)\}_{j=1}^N$. For each $\epsilon < \epsilon_0$, we may assume that the numbers

$$b_j(s) := \sup_{B(p_j,\epsilon)} \{|e(f_s)|\}$$

are unbounded for all j. Shrinking ϵ if necessary, we may assume that $B(p_j, 2\epsilon)$ are all disjoint and set

$$a_j := \lim_{\epsilon \to 0} \limsup_{s \to \infty} \int_{B(p_j,\epsilon)} \| e(f_s)| - |e(f_0)\| \omega, \tag{4.5}$$

the energy loss at each p_j at $s = \infty$ when extending the domain of f_0 over p_j's. f_0 and these a_j clearly satisfy (4.4), and it remains to show that $a_j \geq B_0$. That is, the energy loss is at least the minimum requirement for non-constant holomorphic map.

The inequality is verified by standard renormalization of f_s near each bubble point by techniques of Sacks–Uhlenbeck type. Let $\bar{p}_j^s \in B(p_j, \epsilon)$ be the point at which maximum energy density is achieved: $|e(f_s)(\bar{p}_j^s)| = b_j(s)$. Then $\bar{p}_j^s \to p_j$ as $s \to \infty$ (up to subsequences, by compactness of Σ). Fixing an appropriate holomorphic coordinate, we rescale the geodesic ball $B(p_j, \epsilon)$ to define the renormalization

$$\tilde{f}_s : B(0, \epsilon b_j(s)) \to Z$$

by

$$\tilde{f}_s(y) = f_s\left(\bar{p}_j^s + \frac{y}{b_j(s)}\right). \tag{4.6}$$

Straightforward computations show that $|e(\tilde{f}_s)| \leq 1$ and $|e(\tilde{f}_s)(0)| = 1$. Via stereographic projection, we may regard each $\tilde{f}_s$ as a map on a domain of $\mathbb{S}^2\backslash\{p^+\}$, where p^+ is the north pole, or point of infinity. Since $b_j(s) \to \infty$ as $s \to \infty$, the domains approach the punctured disc as $s \to \infty$. The conditions for energy densities remain true on the sphere due to conformal invariance. With the required energy bound, we may then apply part (a) to extract a subsequence converging to a jJ-holomorphic map $\tilde{f}_{p_j} : \mathbb{S}^2\backslash\{p^+\} \to Z$ as $s \to \infty$. Removing the singularity by Theorem 4.2 we obtain the "bubble map" $\tilde{f}_{p_j} : \mathbb{S}^2 \to N$. Since $e(\tilde{f}_s)(0) \neq 0$, it is not a constant map and therefore $E(\tilde{f}_{p_j}) \geq B_0$ by Theorem 4.3. By Fatou's lemma and (4.5), we have $a_j \geq E(\tilde{f}_{p_j})$, and therefore $a_j \geq B_0$. □

A drawback of the renormalization procedure above is that the inequality $a_j \geq E(\tilde{f}_{p_j})$ may be strict. That is, the bubble map might not capture all the energy loss a_j when extending f_0 over bubble point. The main reason for this is that the rescaling factor $b_j(s)$ in (4.6) may be too large so that too much energy is pushed toward the north pole p^+, forming a *connecting tube* with positive energy. The tube is removed when extending $\tilde{f}_{p_j}$ over p^+, resulting in energy loss. The loss is certainly undesirable as it creates topological jump in the limit, making continuity more difficult. The situations are illustrated in the next page. Figure 1 illustrates the renormalization at s and Fig. 2 illustrates the limit $s \to \infty$ with a connecting tube of positive energy to be removed.

To avoid such setback, we adjust the rescaling factor so that the energies on the tubes are controlled by ϵ. Since energies of non-constant holomorphic maps may not

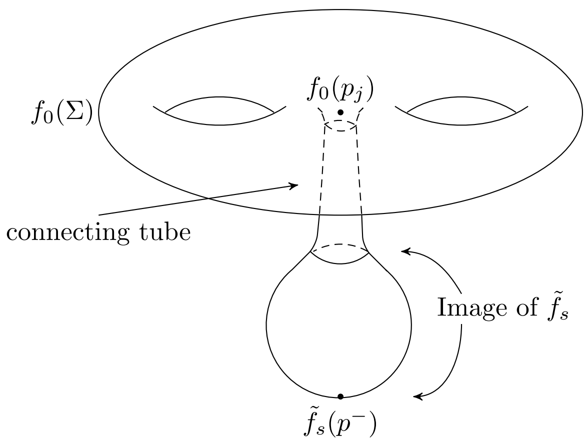


Fig. 1. Formation of connecting tube.

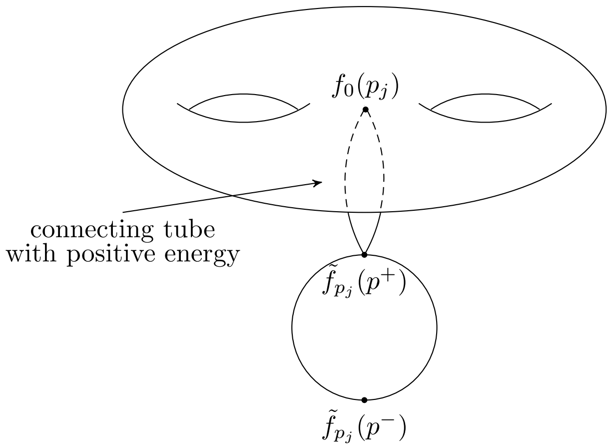


Fig. 2. Bubbling with energy loss.

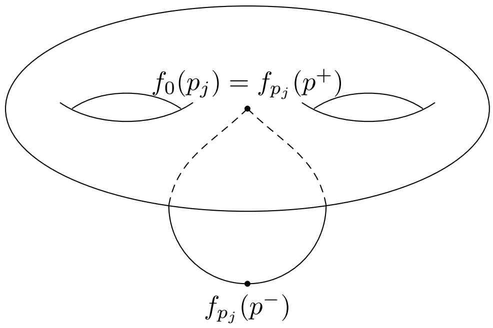


Fig. 3. Bubbling without energy loss.

be arbitrarily small, they must approach 0 as $\epsilon \to 0$ (cf. [17, Sec. 5]). The modified process introduces the "bubble tree" description of moduli space at infinity, which associates a sequence of $\mathbb{S}^2$ and corresponding holomorphic maps on $\mathbb{S}^2$ at a bubble point. These bubbles in total will preserve all the original energy and we expect bubbles to be attached as illustrated in Fig. 3.

For simplicity, we omit the subscript j of the bubble point p_j, as well as all the data associated to it, in describing the energy preserving bubbling at p_j.

Theorem 4.7 (Bubble Tree). *For each bubble point p, there exists a finite sequence of holomorphic maps $\{\tilde{f}_p^l\}$ from $\mathbb{S}^2$ to Z so that*

$$\sum_l E(\tilde{f}_x^l) = a. \tag{4.7}$$

The general principle for adjusted renormalization is that for each ϵ and neighborhood $D(p, \epsilon)$ around p, we pull back the maps by some carefully designed conformal maps so that they are all defined on domains of two sphere $\mathbb{S}^2_p$. Then we apply renormalization techniques of Theorem 4.6 on sphere with possibly new bubble points on $\mathbb{S}^2_p$. On these points, we iterate the construction, forming "bubbles on bubbles", or a *bubble tree*. The technical designs here ensure that the energy loss, or the energy of connecting tubes, are controlled by ϵ, which vanishes as $\epsilon \to 0$.

Proof. Fix $\epsilon > 0$. Each disc $B(p,\epsilon)$ in the proof of the previous theorem corresponds to a domain $B_\epsilon \subset \mathbb{S}^2_p$ under stereographic projection. Denote the north pole and south pole of $\mathbb{S}^2_p$ by p^+ and p^-, respectively. The bubble point p then corresponds to the south pole $p^- \in B_\epsilon$. Let also

$$a(\epsilon, s) = \int_{B_\epsilon} \| e(f_s)| - |e(f_0)\| \omega, \tag{4.8}$$

be numbers so that

$$\lim_{\epsilon\to 0} \limsup_{s\to\infty} a(\epsilon, s) = a \geq B_0$$

as in Theorem 4.6.

We consider the following composition of conformal maps:

$$R_{\epsilon,s} : \mathbb{S}^2_p \xrightarrow{\rho_{t_{\epsilon,s}}} \mathbb{S}^2_p \xrightarrow{T_{\epsilon,s}} \mathbb{S}^2_p \xrightarrow{\sigma} T_p\Sigma \xrightarrow{exp} \Sigma. \tag{4.9}$$

Here, exp is the exponential map with respect to complex structure j and σ is an orientation preserving conformal map from $\mathbb{S}^2$ to $\mathbb{R}^2$ such that $\sigma(p^-) = 0$ and $\sigma(p^+) = \infty$ (e.g., a stereographic projection). $T_{\epsilon,s}$ is a conformal transformation on $\mathbb{S}^2_p$ corresponding to the translation of $T_p\Sigma$ that translates the center of mass of the measure $\| e(f_s)| - |e(f_0)\|$ to z-axis. Finally, $\rho_{t_{\epsilon,s}}$ is the conformal transformation corresponding to radial dilation of $T_x\Sigma$ by $t_{\epsilon,s} > 0$. We give a qualitative description of the scale $t_{\epsilon,s}$ below.

Let $C_0 > 0$ be a constant less than $\frac{B_0}{2}$, where B_0 is the lower bound of energies of non-constant holomorphic maps described in Theorem 4.3. The scale $t_{\epsilon,s}$ is chosen so that

$$\int_{B^s_\epsilon \setminus H^-} \| e(f_s)| - |e(f_0)\| = C_0, \tag{4.10}$$

where H^- is the southern hemisphere and $B^s_\epsilon := R^*_{\epsilon,s} B(p,\epsilon)$. In another words, a constant amount of energy is retained on the northern hemisphere throughout the process. Such a scale is possible by continuous dilation to continuously spread out the energy concentration at p (or p^-). Apparently, $t_{\epsilon,s}$ is introduced to control the amount of energy pushed toward ∞, or p^+. Nevertheless, it is still necessary that $t_{\epsilon,s} \to \infty$ as $s \to \infty$. Indeed, for each s, let $\bar{p}^s \in B(p,\epsilon)$ be the point where $|e(f_s)|$ achieve its supremum. These points converge to the bubble point p as $s \to \infty$ and energies of $|e(f_s)|$ are arbitrarily concentrated near p^-. Therefore, for (4.10) to be true, the scaling factors $t_{\epsilon,s}$ must be large enough so that $t_{\epsilon,s}\bar{p}^s \nrightarrow p$, which requires $t_{\epsilon,s} \to \infty$ so that a constant amount of energy to be kept away from p^-. The renormalized map is then

$$\tilde{f}_{\epsilon,s} := R^*_{\epsilon,s}(f_s|_{B(p,\epsilon)}), \tag{4.11}$$

which are holomorphic with respect to the pull back complex structures $j_s = R^*_{\epsilon,s}J$. The structures j_s approach the standard complex structure on $\mathbb{S}^2_p$ as $s \to \infty$. Since $t_{\epsilon,s} \to \infty$ as $s \to \infty$, the domains of $\tilde{f}_{\epsilon,s}$ approach $\mathbb{S}^2_p \backslash \{p^+\}$ as $s \to \infty$. Moreover, with conformal invariance, we have $E(\tilde{f}_{\epsilon,s}) \leq C$,

$$|E(\tilde{f}_{\epsilon,s}) - E(\tilde{f}_{\epsilon,0})| > \frac{B_0}{2}, \tag{4.12}$$

and

$$\int_{H^+} \| e(\tilde{f}_{\epsilon,s})| - |e(\tilde{f}_{\epsilon,0})\| = C_0, \tag{4.13}$$

where H^+ is the northern hemisphere and $\tilde{f}_{\epsilon,0} = \lim_{s\to\infty}(R^*_{\epsilon,s}f_0)$. We may repeat the arguments in the proof of Theorem 4.6 and obtain the list of bubbling points

$$\mathbb{B}_\epsilon := \{y_{1,\epsilon}, \ldots, y_{l,\epsilon}, p^+\} \subset \mathbb{S}^2_x \tag{4.14}$$

and a jJ-holomorphic map $\tilde{f}_{\epsilon,p} : \mathbb{S}^2_p \to Z$ so that $\tilde{f}_{\epsilon,s} \to \tilde{f}_{\epsilon,x}$ in C^1 on $\mathbb{S}^2_p \backslash \mathbb{B}_\epsilon$. The adjusted renormalization satisfies part (b) of Theorem 4.6: For all ϵ,

$$e(\tilde{f}_{\epsilon,s}) \to e(\tilde{f}_{\epsilon,x}) + \sum_{j=1}^{l} a_{j,\epsilon}\delta(y_{j,\epsilon}) + \tau_{\epsilon,p}\delta(p^+) \tag{4.15}$$

as $s \to \infty$, where $\tau_{\epsilon,p}$ is the energy loss at infinity.

Finally, we shrink the radius ϵ of the initial disc around p. Pick a sequence $\epsilon_s \to 0$ as $s \to \infty$. The bubble points and associated energy losses

$$\{y_{1,\epsilon_s}, \ldots, y_{l,\epsilon_s}, a_{1,\epsilon_s}, \ldots, a_{l,\epsilon_s}, \tau_{\epsilon_s,p}\}_s, \tag{4.16}$$

range in compact sets and possess subsequences converging to

$$\mathbb{B} := \{y_1, \ldots, y_l, a_1, \ldots, a_l, \tau_p\} \tag{4.17}$$

as $s \to \infty$. The corresponding holomorphic maps $\tilde{f}_{\epsilon_s,p}$ converge to a map $\tilde{f}_p$: in C^1 on $\mathbb{S}^2_p \backslash \mathbb{B}$ which extends to the entire $\mathbb{S}^2_p$ holomorphically. Since f_0 is smooth on $B(p,\epsilon)$, by conformal invariance we have $e(\tilde{f}_{\epsilon,0}) \to 0$ as $\epsilon \to 0$ in measure. Consequentially, we have

$$e(\tilde{f}_{\epsilon_s,s}) \to e(\tilde{f}_p) + \sum_{n=1}^{l} a_n\delta(y_n) + \tau_p\delta(p^+) \tag{4.18}$$

and

$$\int_{H^+} |e(\tilde{f}_{\epsilon_s,s})| \to C_0 \tag{4.19}$$

as $s \to \infty$.

Next we observe that τ_p in (4.18) is controlled by ϵ. Fix a small disc Ω centered at p^+ of radius ϵ small enough so Ω contains no other bubble point. Since the energy of $\tilde{f}_p$ on H^+ is no greater than $C_0 < B_0$, the minimum requirement for non-constant holomorphic map. The homology class, $\alpha_{\tilde{f}_p}$, of $\tilde{f}_p$ on Ω is therefore trivial. By (4.3) in Proposition 4.5, we have

$$\tau_p \leq \text{Area}(\tilde{f}_p(\Omega)) \leq C \quad \text{length}^2(\tilde{f}_p(\partial\Omega)), \tag{4.20}$$

and the right-hand side is dominated by ϵ_s by continuity of $\tilde{f}_p$. Let $s \to \infty$, or $\epsilon_s \to 0$, we therefore eliminate energy loss and have the conservation of energy

$$a = \sum_{n=1}^{l} a_n. \tag{4.21}$$

The energy-preserving renormalization may be iterated. Near each $y_n \in \mathbb{B}$, we may renormalize $\tilde{f}_p$ to obtain a collection of bubble points and bubble energies

$$\mathbb{B}_n^1 := \{y_{n,1}, \ldots, y_{n,l_n}, a_{n,1}, \ldots, a_{n,l_n}\}$$

on another sphere $\mathbb{S}^2_{y_n}$ equipped with a "bubble" $\tilde{f}_{y_n} : \mathbb{S}^2 \to N$ constructed identically as above. Of course, these new energies satisfy

$$a_n = \sum_{t=1}^{l_n} a_{n,t}.$$

The process continues and we end up with Σ and a collection of bubbles wedged at various bubble points, forming a bubble tree. □

The origins of new bubbles $y_{1,\epsilon}, \ldots, y_{l,\epsilon}$ in (4.14), as well as their ϵ limit (4.17), can be explicitly explained. The only bubble point on $B(p,\epsilon)$, namely p, forms due to the accumulation of $\bar{p}^s$, points where $|e(f_s)|$ achieves supremum. The renormalized map $\tilde{f}_{\epsilon,s}$ then has maximum energy density at $R^{-1}_{\epsilon,s}(\bar{p}^s)$. Therefore, the only possibilities for new bubbles to form upon the next renormalization is the presence of new limit points from the sequence $\{R^{-1}_{\epsilon,s}(\bar{p}^s)\}$ in $\mathbb{S}^2_p$. Since the finite energy condition is invariant under $R_{\epsilon,s}$, there are only finitely many such points $y_{1,\epsilon}, \ldots, y_{l,\epsilon}$ and so are their ϵ-limit points and corresponding bubbles.

It is important to point out that the limiting map $\tilde{f}_p : \mathbb{S}^2_x \to Z$ may itself be a constant map and $E(\tilde{f}_p) = 0$. That is, the energy of the bubble is entirely concentrated at the new bubble point(s) $\mathbb{B}$ on it and therefore entirely bubbled off upon renormalization. We refer to such sphere as a *ghost bubble*. A ghost bubble for the non-constant holomorphic map $\tilde{f}_p$, however, must contain at least 2 new bubble points.

Lemma 4.8 (cf. [17, Lemma 4.2]). *For $\tilde{f}_p$ in Theorem* 4.7, *if $E(\tilde{f}_p) = 0$, the integer l in $\mathbb{B}$ is at least* 2.

It follows from this lemma, Theorems 4.6 and 4.7 that the new bubbles attached to a ghost bubble has maximum energy at least C_0 less than the ghost bubble before renormalization. Therefore, each new level of bubble tree, ghost or not, consists of bubbles with energies no grater than the energy of the root bubble minus C_0. Consequentially, the bubble tree is of finite length with a finite number spheres.

All these newly induced objects extend both the domain of original maps and give rise to a new map $\tilde{f}$ on the extended domain consisting of a finite number of $\mathbb{S}^2$'s wedging at bubble points. We denote the union of these spheres by $\bar{\mathcal{T}}$. A more geometrical description of $\bar{\mathcal{T}}$ is through a tower of $\mathbb{S}^2$-fibrations. A complex structure on Σ determines the complex tangent bundle $T\Sigma := F\Sigma \times_{\mathbb{C}^*} \mathbb{C}$, where $F\Sigma$ is the complex frame bundle. Stereographic projection σ, an orientation preserving conformal map $\mathbb{S}^2 \to \mathbb{R}^2$, compactifies $T\Sigma$ into $S\Sigma := F\Sigma \times_{\mathbb{C}^*} \mathbb{S}^2$. The newly formed bubble points $\{y_1, \ldots, y_l\}$ are therefore elements in the fiber of $S\Sigma$ over bubble point p. Renormalization on y_n's therefore give rise to more bubble points on $SS\Sigma$, and so forth. We therefore have a tower of sphere fibrations

$$\cdots \to S^m\Sigma \to S^{m-1}\Sigma \to \cdots \to S\Sigma \to \Sigma. \tag{4.22}$$

Definition 4.9 ([17]). A bubble domain B at level m is a fiber ($\simeq \mathbb{S}^2$) at the level m in (4.22). A bubble domain tower is an extension of Σ by a finite union of bubble domains:

$$\bar{\mathcal{T}} := \Sigma \bigcup_{m=1}^{N} T_m, \tag{4.23}$$

where each T_m is a finite collection of bubble domains at level m. A *bubble tree* with bubble domain $\mathcal{T}$ is $\bar{\mathcal{T}}/\sim$, identifying the north pole of each bubble domain (fiber) with its base point.

$\mathcal{T}$ has a natural tree structure, where vertices are maps and bubble points form the incident edges. The discussions above associate to every sequence of holomorphic maps a unique bubble tree $\mathcal{T}$ where Σ and every sphere is mapped holomorphically into Z. The image of $\mathcal{T}$ under $\tilde{f}$ is known as a cusp curve ([22]).

Gromov compactness is constructed in this extended scope. For each s, we may extend the domain of f_s from Σ to $\mathcal{T}$ by appropriate surgeries. The precise definition requires the following lemma.

Lemma 4.10 ([17, Lemma 6.1] on *Extension*). *For each $A > 0$, there is $\epsilon_A > 0$ such that for all $\epsilon < \epsilon_A$ and continuous $L^{1,2}$ map*

$$f : \Sigma \backslash B(x, \epsilon) \to N$$

with $E(f) < A$ extends to a continuous $L^{1,2}$ map $\bar{f} : \Sigma \to N$.

Moreover, the following estimate hold on $B_f = B(x, r_f)$ with $\epsilon < r_f < \sqrt{\epsilon}$:

$$\int_{B_f} |d\bar{f}|^2 \le C|\log \epsilon|^{-1}$$

and

$$dist(\bar{f}(y), f(z)) \le C|\log \epsilon^{-\frac{1}{2}}| \quad \forall\, y \in B_f, \quad z \in \partial B_f.$$

With the lemma, we extend the domain of f_s, or construct the *prolongation* of f_s on $\mathcal{T}$. For each holomorphic $f_s : \Sigma \to Z$ considered above, we define

$$\mathcal{P}_\epsilon(f_s) : \mathcal{T} \to Z$$

separately on Σ and the spheres as follows.

First restrict f_s on $\Sigma \backslash \cup B(p, \epsilon)$ for ϵ small enough, and extend f_s across the small discs around bubble points to a map $\bar{f}_s$ according to Lemma 4.10. We define

$$\mathcal{P}_\epsilon(f_s)(z) = \bar{f}_s(z); \quad \forall\, z \in \Sigma.$$

Let now $z \in \mathbb{S}^2_p$, a bubble attached to a bubble point $p \in \mathcal{T}$. Each $\epsilon > 0$ is associated to a disc $B_\epsilon = B(p^+, \epsilon)$ around the north pole so that for large enough s, the renormalized $\tilde{f}_{\epsilon,s}$ is defined outside D_ϵ. The map $\tilde{f}_{\epsilon,s}$ has its bubble points $\{y_j\}$, and let $B_j := B(y_j, \epsilon)$. We then restrict $\tilde{f}_{\epsilon,s}$ on $\mathbb{S}^2 \backslash \cup_j B_j$ and extends again by Lemma 4.10 to $\hat{f}_p : \mathbb{S}^2_p \to Z$. We define

$$\mathcal{P}_\epsilon(f_s)(z) = \hat{f}_p(z); \quad \forall\, z \in \mathbb{S}^2_p.$$

Identical definitions apply on neighborhoods around other bubble points. We then have the rigorous sense of Gromov compactness of sequence of holomorphic maps with bounded energies.

Theorem 4.11 ([17, Theorem 6.2] on *Gromov Compactness*). *Let $\{f_s\}$ be a sequence of jJ-holomorphic maps $\Sigma \to Z$. Then there is a bubble tree $\mathcal{T}$, and a sequence $\epsilon_s \searrow 0$ as $s \to \infty$ such that a subsequence of*

$$\mathcal{P}_{\epsilon_s}(f_s) : \mathcal{T} \to Z$$

converges in $C^0 \cap L^{1,2}$ to a jJ-holomorphic map $f : \mathcal{T} \to Z$. The convergence is in $C^r(K)$ for all compact set K away from bubble points.

We say that such f_s *Gromov converge* to f on the bubble tree $\mathcal{T}$.

5. Bubbling of Vortex Moduli Spaces

We now return to vortex moduli spaces. The bubbling phenomenon we study takes place on a family of vortices $\{[D_s, \phi_s]\} \subset \nu_{k+1,0}(s)$ approaching the boundary of $\nu_{k+1}(s)$ as $s \to \infty$. By Theorem 3.2, each $[D_s, \phi_s]$ projects down to a tuple of linearly equivalent divisors $(E_{s,i})_{i=0}^k \in \mathbb{E}_{k+1,0}$ that approaches the boundary of $\mathbb{E}_{k+1}$. Convergence to the boundary indicates the coalescence of these divisors at infinity, or common zeros of these tuples of sections. Precisely, we write each divisor $E_{s,i}$ into a sum of two divisors

$$E_{s,i} = \sum_{j=1}^{N} a_{ij}(s) p_{ij}(s) + \sum_{l=1}^{N_i(s)} b_{il}(s) q_{il}(s), \tag{5.1}$$

where the integral coefficients above sum up to r. The first sum corresponds to coalescence, that is, $a_{ij}(s) \to a_j$ and $p_{ij}(s) \to p_j$ as $s \to \infty$ for all i. We let

$$E := \sum_{j=1}^{N} a_j p_j.$$

The second sum then consists of points that remain separated as $s \to \infty$. That is, $b_{il}(s) \to b_{il}$, $N_{si} \to N_i$, $q_{il}(s) \to q_{il}$ as $s \to \infty$ and

$$\bigcap_i \{q_{il}\} = \emptyset.$$

The divisors

$$E_{0i} := \sum_{l=1}^{N_i} b_{il} q_{il}$$

give rise to new holomorphic maps with degree $r - \sum_j a_j$. Note that $a_j(s)$, $N_i(s)$, and $b_{il}(s)$ in (5.1) are all integers and may be assumed constants in s.

On the other hand, the vortices $\{[D_s, \phi_s]\} \subset \nu_{k+1,0}(s)$ correspond to degree r maps $\{f_s\} \subset \mathcal{H}_{r,k}$ via the diffeomorphism Φ_s described in Sec. 3:

$$f_s(p) := [\phi_{s,0}(p) : \cdots : \phi_{s,k}(p)].$$

Equip $\mathbb{CP}^k$ with the Fubini–Study metric, we attempt to explicitly express the energy density of f_s. For each s, we observe the energy density $e(f_s)$ defined by

$$e(f_s)\omega = f_s^* \omega_{FS} = \partial\bar{\partial} \log \left(\sum_{i=0}^{k} |\phi_{s,i}|^2 \right). \tag{5.2}$$

f_s's are of uniformly bounded, in fact constant, energies:

$$E(f_s) = \int_\Sigma e(f_s)\omega = \frac{1}{2\pi} r \quad \forall\, s$$

and they fit into the discussions of Sec. 4. In particular, $e(f_s)$'s blow up at finitely many points. Straightforward computations show that the only possible blow up points are $p_1, \ldots, p_N \in supp(E)$. For each j, fix a normal neighborhood $B_j := B(p_j, \epsilon)$ small enough so that $B_j \cap_{s,i} E_{s,i} = \{p_{ij}(s)\}$. Given a local trivialization, each section $\phi_{s,i}$ is locally given by

$$\phi_{s,i} = (z - p_{ij}(s))^{a_j} f_{s,i}, \tag{5.3}$$

where $p_{ij}(s) \to 0$ as $s \to \infty$ and $f_{s,i}$ are non-vanishing holomorphic functions on B_j. Moreover, $f_{s,j}$ converge smoothly to a non-vanishing holomorphic function on B_j by Theorem 3.4. With respect to this trivilization, the globally defined $(1,1)$

form (5.2) is then locally given by

$$
\begin{aligned}
f_s^*\omega_{FS} &= e(f_s)dz\wedge d\bar z\\
&= \frac{\sqrt{-1}}{2\pi}\left(\frac{\sum_{i=0}^k |z-p_{ij}(s)|^{2a_j-2}G_{s,i}\sum_{i=0}^k |z-p_{ij}(s)|^{2a_j}|f_{s,i}|^2}{\left[\sum_{i=0}^k |z-p_{ij}(s)|^{2a_j}|f_{s,i}|^2\right]^2}\right)dz\wedge d\bar z\\
&\quad -\frac{\sqrt{-1}}{2\pi}\left(\frac{\begin{array}{c}\sum_{i=0}^k |z-p_{ij}(s)|^{2a_j-2}(z-p_{ij}(s))F_{s,i}\\ \times\sum_{i=0}^k |z-p_{ij}(s)|^{2a_j-2}\overline{(z-p_{ij}(s))}H_{s,i}\end{array}}{\left[\sum_{i=0}^k |z-p_{ij}(s)|^{2a_j}|f_{s,i}|^2\right]^2}\right)dz\wedge d\bar z, \qquad (5.4)
\end{aligned}
$$

where $F_{s,i}, G_{s,i}$, and $H_{s,i}$ are smooth and non-vanishing functions on B_j consisting of $f_{s,i}$ and its derivatives. They converge in C^∞ and the only sources of singularities are $|z-p_{ij}(s)|$'s. It is then clear that the bubbling behaviors depend crucially on the convergence behaviors of $p_{ij}(s)$ to 0 as $s\to\infty$. We first observe the outcome when singularities are formally ignored:

Theorem 5.1 (Formal Removal of Singularities). *Fix a Hermitian line bundle (L,H) over Σ. Given a sequence of vortices $\{[D_s,\phi_s]\}\subset\nu_{k+1,0}(s)$ approaching the boundary of $\nu_{k+1}(s)$, there exists a finite set of points $\{p_1,\dots,p_N\}\subset\Sigma$, integers $\{a_1,\dots,a_N\}\subset\mathbb{N}$ such that $\sum_j a_j\le r$, and vortices $[D_s',\phi_s']$ with smooth (subsequential) limit $[D_0,\phi_0]$ on line bundle*

$$L_0 := L\otimes_j \mathcal{O}(-a_jp_j),$$

such that

- $[D_s',\phi_s'] = [D_s,\phi_s]$ *on* $\Sigma\backslash\{p_1,\dots,p_N\}$ *(via the isomorphism* $L_0\simeq L$ *on* $\Sigma\backslash\{p_1,\dots,p_N\}$*).*
- D_s' *and* ϕ_s' *satisfy the vortex equation*

$$\begin{cases} D_s'^{(0,1)}\phi_{s,i}' = 0 \quad \forall\, i\\ \sqrt{-1}\Lambda F_{D_s'} + \dfrac{s^2}{2}\left(\displaystyle\sum_{i=0}^k |\phi_{s,i}'|_H^2 - 1\right) = 0,\end{cases}\qquad (5.5)$$

  on $L_0\to\Sigma$.
- $[D_0,\phi_0]$ *satisfies*

$$\begin{cases} D_0^{(0,1)}\phi_{0,i} = 0 \quad \forall\, i\\ \displaystyle\sum_{i=0}^k |\phi_{0,i}|_H^2 - 1 = 0,\end{cases}\qquad (5.6)$$

  on $L_0\to\Sigma$.

Proof. We continue the usage of notations introduced in (5.1)–(5.4).

For large enough s, a family $\{[D_s, \phi_s]\}$ is uniquely associated with a family of tuples $\{(E_{s,0}, \ldots, E_{s,k}, \tau_s)\}$, where $\tau_s \in (\mathbb{C}^*)^{k+1}/\mathbb{C}^*$ as in Theorem 3.2 and $E_{s,i}$ as in (5.1):

$$E_{s,i} = \sum_{j=1}^{N} a_j p_j^s + \sum_{l=1}^{N_i} b_{il} q_{il}^s.$$

Let ψ_j be the defining meromorphic section of $\mathcal{O}(-a_j p_j)$ and consider the holomorphic sections $\phi_s' = (\phi_{s,i}')_i$, defined by

$$\phi_{s,i}' := \phi_{s,i} \otimes_j \psi_j \in H^0(\Sigma, L_0). \tag{5.7}$$

Choose $\psi = 1$ away from p_j's so that on every compact subset $K \subset \Sigma \backslash \{p_1, \ldots, p_N\}$, $L|K \simeq L_0|K$ and $\phi_{s,i}' = \phi_{s,i}$. The sections $\phi_{s,i}'$ do not have common zero and define a degree $r - l$ holomorphic map $f_s' : \Sigma \to \mathbb{CP}^k$, where $l = \sum_j a_j$ and $(f_s')^* \mathcal{O}(1) \simeq L_0$ smoothly. $[D_s', \phi_s']$ are then vortices defined by f_s' that satisfy (5.5) via identical construction of Φ_s in (3.1). Since $f_s' = f_s$ on $\Sigma \backslash \{p_1, \ldots, p_N\}$, the first and second statements of the theorem are clear from our constructions.

Staying in the compact region of the moduli space, the limit of $[D_s', \phi_s']$ is then naturally constructed from the limit of f_s'. The energy densities $e(f_s')$ are uniformly bounded on the entire Σ since they are defined by sections whose zeros $\sum_{l=1}^{N_i} b_{il} q_{il}^s$ do not coalesce as $s \to \infty$. Let $f_0 \in \mathcal{H}_{r-l,k}$ be their limit, which defines a vortex $[D_0, \phi_0]$ on L_0. It is clear that $[D_s', \phi_s'] \to [D_0, \phi_0]$ in C^1.

To verify that $[D_0, \phi_0]$ satisfies (5.6), we note that holomorphic condition from first equation of (5.5) continues to hold as $s \to \infty$ via standard elliptic regularity arguments. As for the second equation, we note that $\sqrt{-1}\Lambda F_{D_s'}$ are uniformly bounded in s. Indeed, $F_{D_s'}$ is the pullback of curvature form F_{FS} on $\mathcal{O}(1)$, which is proportional to ω_{FS}, via f_s'. Therefore, $\sqrt{-1}\Lambda F_{D_s'} = Ce(f_s')$ for all s, which are uniformly bounded by the discussions above. Dividing the second equation of (5.5) by $\frac{s^2}{2}$ and let $s \to \infty$, the proof is completed. □

Evidently, formal removal of singularities reduces the topological degree of L by $l = \sum_j a_j$. The loss is a clear consequence of concentration of energy densities of f_s. In another words, curvature forms corresponding to f_s approach a smooth form plus a Dirac delta current supported on these isolated singularities, and the extension simply ignore the singular part. These vortices $[D_s', \phi_s']$ corresponds to the map defined by

$$f_s'(z) = [f_{s,0}(z) : \cdots : f_{s,k}(z)]$$

from (5.3), $e(f_s') = W_s$ converges to the smooth function $W_0 = e(f_0)$ as $s \to \infty$.

As in Theorems 4.6 and 4.7, we apply the carefully designed renormalization process to smooth out the singularities and use the limiting map to obtain the limiting vortex preserving the energy ignored in Theorem 5.1 by bubbles. Part of the following theorem is in fact a specialization of observations from [10].

Theorem 5.2 (Renormalization). *For each p_j in Theorem 5.1, there exists $\epsilon_s \to 0$ as $s \to \infty$ so that the geodesic disc $B(p_j, \epsilon_s)$ is conformally equivalent to $B_s \subset \mathbb{S}^2$, an increasing family of domains with $\cup_s B_s = \mathbb{S}^2\backslash\{p^+\}$, and the followings hold:*

- *The pullbacked vortices $[D_s^*, \phi_s^*]$ on B_s, satisfies*

$$\begin{cases} D_s^{*0,1}\phi_{s,i}^* = 0 \\ \sqrt{-1}\Lambda_s^* F_{D_s^*} + \dfrac{s^2}{2t_j(s)^2}\left(\displaystyle\sum_{i=0}^k |\phi_{s,i}^*|_H^2 - 1\right) = 0 \end{cases} \tag{5.8}$$

  on line bundle $L_s^* := R_j(s)^* L$ over B_s. They coincide with the vortices defined by holomorphic functions $\tilde{f}_s := R_j(s) \circ f_s : B_s \to \mathbb{CP}^k$ in the way of (3.1). Conformal maps $R_j(s)$ and parameters $t_j(s) \to \infty$ have been introduced in the proof of Theorem 4.7. (Here, we denote $R_{\epsilon_s,s}, t_{\epsilon_s,s}$ by $R_j(s)$ and $t_j(s)$.)
- *Exactly one of the followings holds true:*
  (a) *There exists a C^1_{loc}-convergent subsequence of $\{[D_s^*, \phi_s^*]\}$ whose limit $[D_j, \phi_j]$ satisfies*

$$\begin{cases} D_j\phi_{j,i} = 0 \quad \forall\, i \\ \displaystyle\sum_{i=0}^k |\phi_{j,i}|_H^2 - 1 = 0 \end{cases} \tag{5.9}$$

  defined on the entire $\mathbb{S}^2$. That is, a holomorphic sphere in $\mathbb{CP}^k$ bubbles off.
  (b) *There exsits points $\{p_j^1, \ldots, p_j^{N_j}\} \subset \mathbb{S}^2$, non-negative integers $a_j^0, a_j^1, \ldots, a_j^{N_j}$, and a C^1_{loc}-convergent subsequence of $\{[D_s^*, \phi_s^*]\}$ on $\mathbb{S}^2\backslash\{p_j^1, \ldots, p_j^{N_j}, p^+\}$, whose limit $[D_j, \phi_j]$ satisfies*

$$\begin{cases} D_j\phi_{j,i} = 0 \quad \forall\, i \\ \sqrt{-1}\Lambda^* F_{D_j} + \dfrac{1}{2}\left(\displaystyle\sum_{i=0}^k |\phi_{j,i}|_H^2 - 1\right) = 0 \end{cases} \tag{5.10}$$

  on a degree a_j^0 line bundle L_j over $\mathbb{S}^2\backslash\{p_j^1, \ldots, p_j^{N_j}, p^+\}$. Moreover, (L_j, D_j, ϕ_j) is the C^1 limit of (L_s^*, D_s^*, ϕ_s^*).
- *On $\mathbb{S}^2$, $\sqrt{-1}\Lambda^* F_{D_j}$ is a distribution given by a smooth function plus $\sum_{l=1}^{N_j} a_j^l \delta(p_j^l)$.*

We make a brief digression to observe the relationship between the scales $t_j(s)$ and zeros $p_{ij}(s)$ for the simple case when $B(p_j, \epsilon)$ is Euclidean. Moreover, we assume that all $f_{s,i}$'s, the non-vanishing parts of the sections in (5.3), are 1:

$$\phi_{s,i} = (z - p_{ij}(s))^{a_j} \tag{5.11}$$

and $f_s(z) = [(z - p_{oj}(s))^{a_j} : \cdots : (z - p_{kj}(s))^{a_j}]$. The extension map f_0 in Theorem 4.2 is therefore constant and of zero energy density. The energy density is simplified considerably:

$$
\begin{aligned}
e(f_s) &= e(f_s) - e(f_0) \\
&= \frac{\sqrt{-1}}{2\pi} a_j \frac{\sum_{i=0}^k |z - p_{ij}(s)|^{2a_j-2} \sum_{i=0}^k |z - p_{ij}(s)|^{2a_j}}{\left(\sum_{i=0}^k |z - p_{ij}(s)|^{2a_j}\right)^2} \\
&\quad - \frac{\sqrt{-1}}{2\pi} a_j \frac{\begin{array}{c}\sum_{i=0}^k |z - p_{ij}(s)|^{2a_j-2}(z - p_{ij}(s)) \\ \times \sum_{i=0}^k |z - p_{ij}(s)|^{2a_j-2}\overline{(z - p_{ij}(s))}\end{array}}{\left(\sum_{i=0}^k |z - p_{ij}(s)|^{2a_j}\right)^2}.
\end{aligned} \tag{5.12}
$$

We denote by $e(f_{s,t})$ the energy density of f_s pullbacked by dilation t. By conformal invariance, we have

$$\int_{B(0,t\epsilon)} e(f_{s,t}) dy_t^2 = a_j(\epsilon, s) \tag{5.13}$$

for all t and $a_j(\epsilon, s) \to a_j$ as $\epsilon \to 0$ and $s \to \infty$. Recall that $t_j(s)$ is the scaling factor so that the masses of $e(f_{s,t})$ are C_0 for all s on the annulus $B(0, t_j(s)\epsilon)\backslash B(0, 1)$ (which correspond to the $\sigma^{-1}(B(0, t_j(s)\epsilon)) \cap H^+$, where σ is the stereographic projection.) Also recall that C_0 is strictly less than half of the lower bound of energies of non-constant holomorphic maps between $\mathbb{S}^2$ and $\mathbb{CP}^k$. Consider

$$
\begin{aligned}
F_s(t) &:= \int_{B(0,1)} e(f_{s,t}) dy_t^2 \\
&= \int_0^{2\pi} \int_0^{\frac{1}{t}} r e(f_s)(r, \theta) dr d\theta.
\end{aligned} \tag{5.14}
$$

The above conditions for t_s then require that

$$F_s(t_s) = a_j(\epsilon, s) - C_0. \tag{5.15}$$

Elementary computations show that

$$t_s = e^{\frac{a_j(\epsilon,s) - C_0 - C}{K_s}}. \tag{5.16}$$

for some $C > 0$ and

$$K_s = \int_0^{2\pi} -e(f_s)|_{\mathbb{S}^1} d\theta. \tag{5.17}$$

Since $e(f_s)$ concentrates near 0, we see that for s large enough, $F_s(1) = C = a_j$. Also note that $t_s \to \infty$ as $s \to \infty$ as expected.

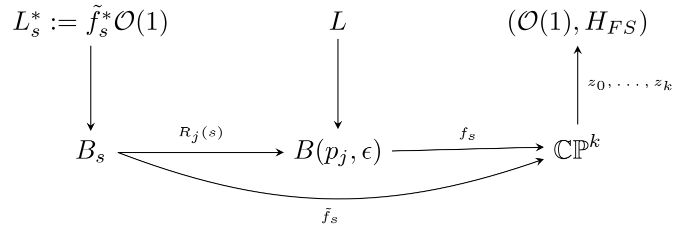
For the general case of metric ω and non-vanishing holomorphic functions $f_{s,i}$, we note that ω_t is asymptotically flat and the functions approach non-vanishing holomorphic functions. (5.16) is then asymptotically satisfied.

We begin the proof of Theorem 5.2.

Proof. With the preparations above, we repeat the proof of Theorem 4.7 except removal of singularity. Recall the composition of conformal transformations in (4.9):

$$R_j(s) : \mathbb{S}^2 \xrightarrow{\rho_{t_j(s)}} \mathbb{S}^2 \xrightarrow{T_s} \mathbb{S}^2 \xrightarrow{\sigma} T_{p_j}\Sigma(\simeq \mathbb{C}) \xrightarrow{exp} B(p_j, \epsilon), \tag{5.18}$$

where T_s and $\rho_{t_j(s)}$ are the conformal transformations on $\mathbb{S}^2$ corresponding to appropriate translations and dilations $y \to \frac{y}{t_j(s)}$ on $\mathbb{C}$, respectively. σ and exp are the usual stereographic projection and exponential map, respectively. Let $\tilde{f}_s := f_s \circ R_j(s)$ and $B_s := R_j(s)^{-1}(B(p_j, \epsilon))$, the following diagram is considered:



(5.19)

The complex structure on B_s is given by

$$J_s(\xi) = j_s(\xi) J_{\mathbb{S}^2}, \tag{5.20}$$

where ξ is the standard complex coordinate on $\mathbb{S}^2$ centered at p^+ and $J_{\mathbb{S}^2}$ is the standard complex structure of the 2 sphere. It is straightforward to check that $j_s \to 1$ uniformly as $s \to \infty$ so that the maps $\tilde{f}_s$ are asymptotically holomorphic with respect to standard complex structure compatible with the round metric.

The vortex equation (1.5) on $L \to B(p_j, \epsilon)$ is then pulled back to (5.8) defined on the left end of the diagram, where the bundle is equipped with background metric $H_s^* := R_j(s)^* H$. The extra factor of $\frac{1}{t_j(s)^2}$ comes from the fact that Λ_s^* is taken with respect to the metric rescaled by $t_j(s)$. The section terms are of course invariant throughout these parametrizations. On the other hand, we may construct solutions $[D_s^\dagger, \phi_s^\dagger]$ to (5.8) directly from holomorphic maps $\tilde{f}_s : B_s \to \mathbb{CP}^k$ in the manner of Φ_s in Sec. 3. Namely, we start with background metric $H_s^\dagger := \tilde{f}_s^* H_{FS}$ on L_s^* and the same holomorphic sections $\phi_{s,i}^* := \tilde{f}_s z_i$ on L_s^*. Turning $H_s^\dagger$ into the special metric $G_s^\dagger$ via a gauge, we obtain the solutions $[D_s^\dagger, \phi_s^\dagger]$. The two solutions, $[D_s^*, \phi_s^*]$ and $[D_s^\dagger, \phi_s^\dagger]$ correspond to two special metrics, G_s^* and $G_s^\dagger$, that are gauge transformed from H_s^* and $H_s^\dagger$, so that the holomorphic pair $(\tilde{f}_s^* \bar{\partial}, \tilde{f}_s^* \phi_s)$ satisfies the second equation of (5.8). G_s^* and $G_s^\dagger$ are therefore determined, up to unitary gauge, by the unique solution to one Kazdan–Warner equation, and we have $[D_s^*, \phi_s^*] = [D_s^\dagger, \phi_s^\dagger]$.

We have identified pullbacked vortices $[D_s^*, \phi_s^*]$ with holomorphic maps $\tilde{f}_s$, we wish to study the convergent behavior of vortices from the Gromov compactness of maps. As we have expected, the existences of bubble point for $\tilde{f}_s$ are determined by the relative rates of convergence of $p_{ij}(s)$ to p_j, and they are observed through the scale $t_j(s)$. We here observe that bubble point exists when $t_j(s)$ grows proportionally to s, in which the scaling gives rise to an *affine* vortex.

- *Case* 1 (*Fast Blow Up*): $\frac{s}{t_j(s)} \to 0$ *as* $s \to \infty$.

  This is a special case of Case 1 of (11.1) in [10]. Here, we can actually observe directly from (5.8). Since $\frac{s}{t_j(s)} \to 0$, the trace of pullbacked curvature approach 0 away from bubble points. In the limit, we obtain a line bundle trivial over $\mathbb{C}\backslash\{\text{bubble points}\}$ with limiting sections $\phi^*_\infty = (\phi^*_{\infty,0}, \dots, \phi^*_{\infty,k})$. Being global holomorphic sections of a trivial line bundle, ϕ^*_∞ defines a holomorphic map from $\mathbb{C}\backslash\{\text{bubble points}\}$, and therefore $\mathbb{S}^2\backslash\{\text{bubble points and } p^+\}$, to $\mathbb{C}^{k+1}$. Removing singularities by Theorem 4.6, we obtain a non-constant holomorphic map form $\mathbb{S}^2$ to $\mathbb{C}^{k+1}$ which violates the asphericality of $\mathbb{C}^{k+1}$. This case is therefore ruled out.

- *Case* 2 (*Slow Blow Up*): $\frac{s}{t_j(s)} \to \infty$ as $s \to \infty$.

  For this extreme, it follows that $\sum_{i=0}^k |\phi^*_{s,i}|^2_H \to 1$ as $s \to \infty$. In another words, for s large enough, we may assume that

$$\bigcap_{i=0}^{k} {\phi^*_{s,i}}^{-1}(\{0\}) = \emptyset$$

and that the limiting sections have no common zero. From (5.4), we see that the holomorphic maps $\{\tilde{f}_s\}$ have uniformly bounded energy densities and do not have bubble point. Let $\tilde{f}_{p_j}$ be its subsequential C^1 limit. The map defines a vortex $[D_j, \phi_j]$, which satisfies (5.9). Indeed, since $e(\tilde{f}^*_s)$, and therefore $\sqrt{-1}\Lambda^*_s F_{D^*_s}$ in (5.8), are uniformly bounded, the curvature term there approaches 0 after dividing by $\frac{s^2}{2(t_j(s))^2}$ and let $s \to \infty$. The holomorphic condition clearly continues to hold as $s \to \infty$ and (5.9) follows. We therefore obtain a finite energy holomorphic map from $\mathbb{C}$ to $\mathbb{CP}^k$, which extends to a map from $\mathbb{S}^2$ to $\mathbb{CP}^k$, or a holomorphic sphere in $\mathbb{CP}^k$.

- *Case* 3 (*Moderate Blow Up*): $\frac{s}{t_j(s)} \to \lambda \in (0, \infty)$ as $s \to \infty$.

  Bubble points occur only in this case, when energy densities blow up roughly proportional to s. With an additional normalization if necessary, we assume that $\lambda = 1$. By conformal invariance of energy, we have $E(\tilde{f}_s) = a_j$ for all s. Following constructions in the proof of Theorem 5.2, let $\{p^1_j, \dots, p^{N_j}_j, p^+\}$ be the bubble points, $\tilde{f}_j$ be the C^1_{loc} limit of $\tilde{f}_s$ on $\Sigma\backslash\{p^1_j, \dots, p^{N_j}_j\}$, and $\bar{f}_j$ be the extension of $\tilde{f}_j$ to $\mathbb{S}^2$. Let $a^l_j \in \mathbb{N}$ be defined as in Theorem 5.2:

$$a^l_j := \lim_{\epsilon\to 0} \limsup_{s\to\infty} \int_{B(p^l_j,\epsilon)} \| e(\tilde{f}_s)| - |e(\bar{f}_j)\| \omega_s. \tag{5.21}$$

The vortex $[D_j, \phi_j]$ defined by $\tilde{f}_j$ therefore satisfies (5.10), where $a^0_j = E(\bar{f}_j)$.

We finally check that the limiting procedure is compatible with the geometric structure of line bundles $L^*_s := \tilde{f}^*_s\mathcal{O}(1)$ over B_s. The degrees of L^*_s are precisely the energies of $\tilde{f}_s$, which is the constant a_j. Each L^*_s is equipped with transition functions $\{\gamma^s_{\alpha\beta}\}$ (pullbacked from those on $\mathcal{O}(1)$ via $\tilde{f}_s$) so that on the overlap, we have the compatibility condition

$$A_{s,\alpha} = -d\gamma_{s,\alpha\beta}(\gamma_{s,\alpha\beta})^{-1} + \gamma_{s,\alpha\beta}A_{s,\beta}(\gamma_{s,\alpha\beta})^{-1}, \tag{5.22}$$

where $A_{s,\alpha}$ and $A_{s,\beta}$ are the connection forms of the unitary connections D_s^* constructed above. That is, the coefficients of these one forms are by definition algebraic expressions $\tilde{f}_s$ and its first derivatives. For a local holomorphic frame u_α, we have

$$A_{s,\alpha} = d' \log H_{s,\alpha}^* = d' \log \left(\frac{|u_\alpha|^2}{\sum_{i=0}^k |\tilde{f}_s(z_i)|^2} \right). \tag{5.23}$$

The C^1_{loc} convergence of $\tilde{f}_s$ therefore imply that A^s_α and A^s_β converge uniformly (in s) on their domains of definitions. Smooth convergence of these transition functions then follow by standard bootstrapping arguments. As $s \to \infty$, (5.22) passes to the limit:

$$A_{j,\alpha} - d\gamma_{j,\alpha\beta}(\gamma_{j,\alpha\beta})^{-1} + \gamma_{j,\alpha\beta} A_\beta (\gamma_{j,\alpha\beta})^{-1} \tag{5.24}$$

on $\mathbb{S}^2 \backslash \{p_j^1, \ldots, p_j^{N_j}, p^+\}$. Therefore, the local transition functions and one forms patch together to give a degree a_j^0 holomorphic line bundle L_j over the punctured sphere. This proves the second statement.

Finally, since the conformal transformations are designed so that there is no energy loss at p^+, and therefore

$$E(\tilde{f}_j) = a_j - \sum_{l=1}^{N_j} a_j^l := a_j^0. \tag{5.25}$$

The curvature current F_{D_j} it defines then satisfies the last statement. □

Iterating the procedure at each bubble point $p_j^l \in \mathbb{S}^2$, the theorem provides an almost complete description on the root of bubble tree. We however still face the hurdle of extending the vortex across p^+, or removal of singularities. Extending holomorphic maps of finite energy over p^+ poses little difficulty by application of Theorem 4.2, but the extensions are not necessarily compatible with the correspondence of maps and vortices. This is the stage where classical extension results for topological fields, such as those in [20, 21], enter. The more general case of critical points to the Yang–Mills–Higgs energy functional, which contains our case, have been discussed in [18]. We briefly summarize the work here.

6. Extension of the Bundle at Infinity and Conical Metrics

6.1. *The smooth case*

Fix a small geodesic disc B_{R_0} around $p+$ within the injectivity radius that contains no other bubble point. The removability of singularities depends on how well one controls certain norms of connection form A and curvature F_A on $B_{R_0} \backslash \{p^+\}$ (cf. [20, 21] for Yang–Mills fields). For critical points to YMH_s, Smith showed that the limiting field may be extended across p^+ if F_A is integrable enough and there is a gauge on which

$$\|A\|_p|_{B_{R_0} \backslash \{p^+\}} \leq O(\rho), \tag{6.1}$$

where $\rho = |\xi|$ is the radial component of the polar coordinate of B_{R_0}. Such an estimate allows one to apply implicit function theorem to solve the required regularity condition

$$d^*(g^{-1}dg + g^{-1}Ag) = 0, \tag{6.2}$$

which ensures the smoothness of extended field on the entire disc B_{R_0}. The establishment of (6.1) has been thoroughly discussed in [18]. It was proved that the bound follows from certain decay condition on the holonomy of the connection D, called the "Condition H":

Definition 6.1 (Condition *H*). Let D be a an affine connection on the bundle L over B_{R_0} and $\gamma_R(t) : [0,1] \to \partial B_{R_0}$ be a smooth positive parametrization of the circle ∂B_{R_0} around p^+. Let $g(R)$ be the holonomy of D over the $\gamma(t)$. That is, for every D-parallel vector field v over $\gamma(t)$, we have

$$v(\gamma(1)) = v(\gamma(0)) \cdot g(R).$$

We say that D satisfies *connection H* if

$$\lim_{R\to 0} g(R) = id, \tag{6.3}$$

pointwise.

This condition is in fact equivalent to the existence of the gauge, on which the angular component of the connection form A decays to 0.

Theorem 6.2 ([18, Theorem 1.1]). *Condition H is equivalent to the existence of a unitary gauge in which*

$$A = A_\rho(\rho,\theta)d\rho + A_\theta(\rho,\theta)d\theta \in \mathfrak{u}(1)$$

with

$$\lim_{\rho\to 0} A_\theta(\rho,\theta) = 0 \tag{6.4}$$

in supnorm topology.

With (6.4), one may apply further gauge transformation, or the "auxiliary gauge" to improve the decay of A.

Theorem 6.3 ([18, Lemma 1.1] on *the Auxiliary Gauge*). *If the connection D satisfies the H condition* (6.4), *there exists a gauge in which $D = d + A$ and the followings hold*:

$$\lim_{\rho\to 0} A_\rho(\rho,0) = 0, \quad \lim_{\rho\to 0} A_\theta(\rho,\theta) = 0, \quad \lim_{\rho\to 0} \frac{\partial}{\partial\rho} A_\theta(\rho,\theta) = 0. \tag{6.5}$$

In the auxiliary gauge, one can estimate the L^p norms of A_ρ and A_θ separately, as in the technical Lemmas 1.2 and 1.3 in [18], so that (6.1) follows. Classical arguments in [20, 21] imply the theorem on removability of singularity:

Theorem 6.4 ([18, Theorem M], Relevant Form). *On $B_{R_0}\backslash\{p+\}$ with Euclidean metric and L a line bundle over it, let A be a connection form satisfying* *condition H, and its curvature form $F \in L^p(B_{R_0})$ be smooth for $p \geq 1$. Assume that (F, ϕ) satisfies the Euler–Lagrange equation of Yang–Mills–Higgs energy functional* (1.4), *and $\phi \in H_2^1(B_{R_0})$. Then, there exists a continuous gauge transformation such that (F, A) is gauge equivalent to a smooth pair $(\tilde{F}, \tilde{\phi})$ over B_4^2 and the bundle extends smoothly to B_{R_0}.*

6.2. *Conical Hermitian metrics*

Unfortunately, the hypotheses of Theorem 6.4 are not guaranteed for the limiting vortex $[D_j, \phi_j]$ constructed in Theorem 5.2. The first source of failure is the integrability of curvature form F_j of D_j. In the notations of diagram (5.19), let again z and ξ be natural complex coordinates of $B(p_j, \epsilon) \subset \Sigma$ and $B_s \subset \mathbb{S}^2$, centered at p_j and p^+, respectively. The Kähler form ω on $B(p_j, \epsilon)$ can be expressed as

$$\omega(z) = \frac{1}{2\pi\sqrt{-1}} g(z) dz \wedge d\bar{z}, \tag{6.6}$$

with $g(0) = 1$. With further holomorphic coordinate change if necessary, the pullback Kähler form is

$$\omega_s^*(\xi) := R_j(s)^* \omega(\xi) = \frac{1}{2\pi\sqrt{-1}} g\left(\frac{\xi}{t_j(s)}\right) \frac{1}{(1 + |\xi|^2)^2} d\xi \wedge d\bar{\xi}, \tag{6.7}$$

and it is clear that ω_s^* approaches the standard round metric as $s \to \infty$. In other words, the limiting vortex the restriction of an affine vortex on $B_{R_0} \backslash \{p^+\} \simeq \mathbb{C} \backslash B(0, R_0')$ for some large R_0'. For such a vortex, the pointwise norm of F_∞^* is controlled by the estimate of energy density given in [24]:

Proposition 6.5 ([24, Corollary 1.4]). *Let $\omega = dx \wedge dy$ be the standard area form on $\mathbb{C}$ and assume that (A, ϕ) is an affine vortex with such that the images of sections have compact closure. Define the energy density by*

$$e(A, \phi) := |F_A|^2 + \sum_{i=0}^{k} |D_A \phi_i|^2 + \frac{1}{4} \left| \sum_{i=0}^{k} |\phi_i|_H^2 - 1 \right|^2.$$

Then for every $\epsilon > 0$ there exists a constant C_ϵ such that:

$$e(A, \phi) \leq C_\epsilon \, |z|^{-4+\epsilon} \tag{6.8}$$

for $|z| \geq 1$.

Pulling back estimate (6.8) onto $\mathbb{S}^2 \backslash \{p^+\} \simeq \mathbb{C}$ as above, the estimate for pointwise norm of F_j we have is

$$|F_j| \leq C_\epsilon' |\xi|^{-2-\frac{\epsilon}{2}} \tag{6.9}$$

for some $\epsilon > 0$. The estimate is clearly insufficient to guarantee any integrability of F_j in the usual Lebesgue measure.

For the vortex $[D_j, \phi_j]$ that does not satisfy the H condition (6.4), a reasonable modification is to associate a *conic* singularity at p^+ to absorb the singularity. We recall the following definition of canical metrics in dimension 2:

Definition 6.6. Let Σ be a Riemann surface and $p \in \Sigma$. A conical Kähler metric of angle β, conical at p, is a metric whose Kähler $(1,1)$-form in a holomorphic coordinate system centered at p looks like:

$$\omega = e^u \frac{dz \wedge d\bar{z}}{|z|^{2-2\beta}}$$

for some $\beta \in (0,1)$ near p. Here $u \in C^0(\Sigma)$.

Such metrics can be realized as the pullback metric of the map $z \to z^\beta$ and also the map $w \to w^{\frac{1}{\beta}}$, where defined, pulls such metrics back to a smooth metric. It is an easy calculation to see that the form of a conic metric at ∞ looks like:

$$\omega^*_{\infty,\beta} := e^u \frac{|\xi|^{-2+2\beta}}{2\pi\sqrt{-1}} \frac{1}{(1+|\xi|^2)^2} d\xi \wedge d\bar{\xi} \tag{6.10}$$

where $\xi = \frac{1}{z}$.

We now recall some properties of the function spaces described by Donaldson [7] and their behavior under the Laplacian (cf. same reference). Introducing a background metric g_β, conical along a divisor D, with associated distance d_β, let:

$$\mathcal{C}^{,\alpha,\beta}(M,D) = \{f \in \mathcal{C}^0(M) \text{ s.t. } \|f\|_{,\alpha,d_\beta} < +\infty\},$$

$$\mathcal{C}_0^{,\alpha,\beta}(M,D) = \left\{f \in \mathcal{C}^{,\alpha,\beta}(M,D) \text{ s.t. } f(x) = 0 \text{ for all } x \in D\right\},$$

where

$$\|f\|_{,\alpha,d_\beta} = \sup_{x\in M} |f(x)| + \sup_{x,y\in M} \frac{|f(x)-f(y)|}{d_\beta(x,y)^\alpha}.$$

As the notation suggests, those spaces depend on β but they are independent of the particular conical metric g_β chosen. This follows from the fact that any two metrics on M which are conical of angle β along D induce equivalent distances on M. We will sometimes refer to functions in these spaces as β-*weighted* functions of a given regularity.

In local holomorphic coordinates z centered at a point of D such that D is the locus $z = 0$ define

$$\Phi(z) = |z|^{\beta-1} z$$

which is clearly a homeomorphism. As noted by Donaldson [7], a function f on M is of class $\mathcal{C}^{,\alpha,\beta}$ if and only if it is $C^{0,\alpha}$ away from D, and $f \circ \Phi^{-1}$ is $C^{0,\alpha}$ for any choice as above of local coordinates around D.

Consider the change of coordinates $z = \psi(w)$, where the map ψ is defined by

$$\psi(w) = w^{\frac{1}{\beta}},$$

with $w = \rho e^{\sqrt{-1}\theta}$, $0 < \theta < \frac{2\pi\beta}{1+\beta}$. Note ψ is a biholomorphism on its image, therefore (w is local holomorphic coordinates around any point not lying in D. One can check that $\Phi \circ \psi$ is bi-Lipschiz, whence it follows that $\mathcal{C}^{,\alpha,\beta}$ is constituted by functions on M that are $C^{0,\alpha}$ away from D, and such that $\psi^* f$ is $C^{0,\alpha}$ for any choice of local (holomorphic) coordinate z around D as above.

Now we pass to recall the definition of one and two forms of class $\mathcal{C}^{,\alpha,\beta}$. A $(1,0)$-form ξ on M is said to be of class $\mathcal{C}^{,\alpha,\beta}$ if it is $C^{0,\alpha}$ away from D, and $\psi^*\xi$ is of class $C^{0,\alpha}$ and it satisfies $\psi^*\xi(\frac{\partial}{\partial w}) \to 0$ as $w \to 0$. Analogously, a $(1,1)$-form η on M is said to be of class $\mathcal{C}^{,\alpha,\beta}$ if it is $C^{0,\alpha}$ away from D and $\psi^*\eta$ is $C^{0,\alpha}$, and both the contractions of $\psi^*\eta$ with $\frac{\partial}{\partial w}$ or $\frac{\partial}{\partial \bar{w}}$ go to zero as $w \to 0$. One then defines:

$$\mathcal{C}^{2,\alpha,\beta} = \{f \in C^2(M\backslash D) \cap C^0(M) \text{ s.t. } f, \partial f, \partial\bar{\partial} f \text{ are of class } \mathcal{C}^{,\alpha,\beta}\}.$$

We also consider the space:

$$\mathcal{H} := \{f \in \mathcal{C}^\infty(M) \text{ where } \Phi^* f \in W^{1,2}\}.$$

Then Donaldson proves the following (cf. [1] for a strengthening).

Theorem 6.7 ([7]). *Let ω be a Kähler metric on the ball $B_6(0)$ which is of class $\mathcal{C}^{,\alpha,\beta}$ and satisfies $a_1\,\Omega \le \omega \le a_2\,\Omega$ for some suitable constants $a_1, a_2 > 0$. Suppose that $\alpha < \mu := \frac{1}{\beta} - 1$ and that f is a function of class $\mathcal{C}^{,\alpha,\beta}$ defined on the ball $B_6(0)$, and $v \in \mathcal{H}$ is a weak solution of the equation $\Delta_\omega v = f$. Then the restriction $v|_{B_1(0)}$ is of class $\mathcal{C}^{2,\alpha,\beta}$.*

With $\beta > \frac{\epsilon}{2}$, we see that $F_j \in L^1_\beta(B_{R_0})$, the space of integrable functions with measure defined by $\omega^*_{\infty,\beta}$ above. Following identical arguments in the proof of [18, Theorem 4.1] (with the condition that ϕ_j is smooth), we conclude that $F_j \in L^p_\beta(B_{R_0})$ for all $p \ge 1$.

Next, we naturally generalize the condition H above with conic metric. A Hermitian metric h on a line bundle is conic if $h(s) = |z|^{2\beta}$ for holomorphic frame s. The associated connection is

$$D = d + i\beta d\theta$$

or, in holomorphic coordinates:

$$d + \beta\frac{dz}{z}.$$

We view this as the standard model.

For $\beta \in (0,1)$, an extra factor of $\rho^{2\beta-2}$ appears in the volume measure and to achieve the decay condition

$$\|A\|_{p,\beta}|(B_{R_0}\backslash\{p^+\}) \le O(\rho), \tag{6.11}$$

the corresponding "H_β condition" for integral estimates to hold true is then

$$\lim_{\rho\to 0} \rho^{2\beta-2} A_\theta(\rho,\theta) = 0. \tag{6.12}$$

The analogous theorem for removal of singularity is then the following, whose proof is entirely identical to Theorem 6.4 except the measure on the L^p space is replaced by the one defined by conic metric. However, with (6.12), all the estimates in [18] remain valid and making substantial use of Theorem 6.7 we have

Theorem 6.8 (Theorem M_β). *On $B_{R_0}\backslash\{p+\}$ with conic metric of angle $2\pi\beta$ at p^+, L a line bundle over it, let A be a connection form satisfying condition H_β, and its curvature form $F \in L^1_\beta(B_{R_0})$ be smooth. Assume that (F, ϕ) satisfies the Euler–Lagrange equation of Yang–Mills–Higgs energy functional* (1.4), *and $\phi \in \mathcal{H}(B_{R_0})$. Then, there exists a continuous gauge transformation such that (F, A) is gauge equivalent to a smooth conic pair $(\tilde{F}, \tilde{\phi})$ over B_{R_0}, the bundle extends smoothly to B_{R_0} and*:

$$\tilde{A} = d + i\beta d\theta + \mathfrak{a}$$

where $\mathfrak{a}$ is smooth.

For the vortex $[D_j, \phi_j]$ in particular, it remains to construct a trivialization in which the connection form of the limiting vortex (D_j, ϕ_j) on $B_{R_0}\backslash\{p^+\}$ satisfies condition H_β.

Theorem 6.9. *For the line bundle L_j and the limiting gauge $[D_j, \phi_j]$ in the second statement of Theorem* 5.2 *failing to satisfy the H condition* (6.4), *we may continuously extended it to $(\mathbb{S}^2, g_\beta)$, where g_β is the conic metric of angle β at the north pole.*

Proof. Let $A_j = A_j(z)dz$, where $A_j(z) \in \mathfrak{u}(1)$. On the domain $\mathbb{C}\backslash B(0, R_0')$, consider, for some $\alpha > 3 - 2\beta > 1$, the differential equation

$$A_j^\gamma = \frac{\partial}{\partial z}\log\gamma + A_j(z) = \frac{\sqrt{-1}}{|z|^\alpha}dz. \tag{6.13}$$

The gauge γ can be easily expressed by integration: (Note that the right-hand side is integrable since $\alpha > 1$.)

$$\gamma = \exp\left(\sqrt{-1}\int_{z_0}^{z}\left(\frac{1}{|z|^\alpha} - A_j(z')\right)dz'\right) \in U(1). \tag{6.14}$$

In polar coordinate (r, θ) of $B_{R'}$, A_γ can be written as

$$A_j^\gamma = e^{\sqrt{-1}\theta}\left(\frac{\sqrt{-1}}{r^\alpha}dr - \frac{1}{r^{\alpha-1}}d\theta\right). \tag{6.15}$$

Pulling back to $\mathbb{S}^2$ via $R_j(s)$ and let $s \to \infty$, we have, in polar coordinate (ρ, θ) of $\mathbb{S}^2$ near p^+, that

$$(A_j^\gamma)^*_\theta := R_j(\infty)^*(A_j^\gamma)_\theta = \rho^{\alpha-1}G(\rho, \theta), \tag{6.16}$$

where G is a smooth non-vanishing function on B_{R_0}. Since $\alpha - 1 - (2 - 2\beta) > 0$, (6.12) holds and we are done. □

6.3. *Bubble trees with cones*

The description of bubble tree at the root level is now complete, and the entire bubble tree is essentially a finite number of iterations of these renormalization processes. To unify the notation, we denote Σ by T_0 and relabel the bubble points by

$$\mathbb{B}^0 := \{p_1^0, \dots, p_{N_0}^0, q_1^0, \dots, q_{N_0'}^0, r_1^0, \dots, r_{N_0''}^0\} \subset T_0, \tag{6.17}$$

where p, q, and r are bubble points where round spheres, conic spheres (raindrops), and holomorphic spheres that are bubbled off respectively. The first level T_1, is then a disjoint union of these $N_0 + N_0' + N_0''$ bubbles with various types:

$$T_1 := \left(\bigsqcup_{i=1}^{N_0} \mathbb{S}^2_{p_i^0}\right) \sqcup \left(\bigsqcup_{j=1}^{N_0'} \mathbb{S}^2_{q_j^0}\right) \sqcup \left(\bigsqcup_{l=1}^{N_0''} \mathbb{S}^2_{r_l^0}\right). \tag{6.18}$$

Each sphere is wedged to its designated bubble point at its north pole, and those spheres from the first two components above may contain new bubble points. We similarly classify them by the types of new bubbles they form, as in $\mathbb{B}^0$:

$$\mathbb{B}^1 := \{p_1^1, \dots, p_{N_1}^1, q_1^1, \dots, q_{N_1'}^1, r_1^1, \dots, r_{N_1''}^1\} \subset T_1. \tag{6.19}$$

$\mathbb{B}^1$ give rise to a new set of $N_1 + N_1' + N_1''$ bubbles whose disjoint union is T_2 with new bubble points $\mathbb{B}_2$. The process is iterated and we have the main theorem of this article:

Theorem 6.10 (Bubble Tree). *The vortices $V_s := \{[D_s, \phi_s]\}$ on a degree r line bundle L over Σ Gromov converge to a vortex $\mathcal{V} := [\mathcal{D}, \Phi]$ over a degree r line bundle $\mathcal{L}$ over a bubble tree $\mathcal{T}$ defined by*

$$\mathcal{T} := T_0 \vee T_1 \vee \cdots \vee T_{N_{\mathcal{V}}}, \tag{6.20}$$

where $T_0 = \Sigma$.

Proof. T_n is constructed inductively as above, with associated bubble points $\mathbb{B}_n$. T_n and T_{n+1} are then wedged at $\mathbb{B}_n \subset T_n$ and the north poles of each sphere in T_{n+1} designated to its bubble point. By Lemma 4.8 and Theorem 4.3, each renormalization reduces the total energy by at least $C_0 > 0$ and therefore the bubble tower consists of at most finite number ($N_{\mathcal{V}}$) of levels.

$\mathcal{L}$ is the line bundle over $\mathcal{T}$ whose restriction to each sphere $\mathbb{S}^2_{p_i^n} \backslash \{p^+\}$ is the holomorphic line bundle determined by the connection $D_{p_i^n}$ that is determined by iterations of Theorem 5.2. Φ is constructed identically. Allowing the possibility of conic singularity, the discussions above conclude that the extensions of vortices are possible whenever the corresponding holomorphic maps extend. Therefore, we may define prolongation of vortices V_s, $\mathcal{V}_s$, identically by their corresponding maps. The Gromov convergence of V_s to $\mathcal{V}$ are then defined by the C^1 convergence from $\mathcal{V}_s$ to $\mathcal{V}$ on $\mathcal{T}$, which follow from convergence of their corresponding maps. Conic singularities are appropriately introduced at points where energy density does not decay fast enough. Since energy is conserved throughout the entire process, the degree of $\mathcal{L}$, or the sum of the energies of $\mathcal{V}$ on Σ and all bubbles, is precisely the original degree r. □

The Gromov limit of a sequence of vortices have been constructed. The natural follow up construction is the moduli space whose boundary includes all these bubble trees. Furthermore, we expect some kind of dynamics, or L^2 metric on the space. The rate of convergence, or equivalently the rate of coalescence of zeros, will play an important role in this metric. We are eager to purse, or learn any possible progress in this direction.

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