Hw 12

§15-6 : ※ 4, 12, 20, 23, 26, 27

§16-1 : ※ 6, 14, 18, 22, 30, 39

§16-2 : ※ 6, 10, 16, 22, 24, 28, 37, 40.

§15-6

※4.

$$f(x,y) = x^2\cos y + y^2 \sin x$$

$$\frac{\partial f}{\partial y} = -x^2 \sin y + 2y \sin x$$

$$\frac{\partial f}{\partial x} = 2x\cos y + y^2 \cos x$$

$$\frac{\partial^2 f}{\partial x^2} = 2\cos y + y^2 \cdot(-\sin x) = \underline{2\cos y - y^2 \sin x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \underline{-2x\sin y + 2y\cos x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \underline{-2x\sin y + 2y\cos x}$$

$$\frac{\partial^2 f}{\partial y^2} = \underline{-x^2\cos y + 2\sin x}$$

12. $f(x,y,z) = \sin(x+z^y)$

$$\frac{\partial f}{\partial x} = \cos(x+z^y)\cdot\frac{\partial(x+z^y)}{\partial x} = \cos(x+z^y)\cdot 1 = \cos(x+z^y)$$

$$\frac{\partial f}{\partial y} = \cos(x+z^y)\cdot\frac{\partial(x+z^y)}{\partial y} = \cos(x+z^y)\cdot z^y\cdot \ln z$$

$$\frac{\partial f}{\partial z} = \cos(x+z^y)\cdot\frac{\partial(x+z^y)}{\partial z} = \cos(x+z^y)\cdot y\cdot z^{y-1}$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x+z^y)\cdot\frac{\partial(x+z^y)}{\partial x} = -\sin(x+z^y)\cdot 1 = \underline{-\sin(x+z^y)}$$

$$\frac{\partial^2 f}{\partial y\partial x} = -\sin(x+z^y)\cdot\frac{\partial(x+z^y)}{\partial y} = \underline{-\sin(x+z^y)\cdot z^y\cdot\ln z}$$

$$\frac{\partial^2 f}{\partial z\partial x} = -\sin(x+z^y)\cdot\frac{\partial(x+z^y)}{\partial z} = \underline{-\sin(x+z^y)\cdot y\cdot z^{y-1} = \frac{\partial^2 f}{\partial x\partial z}}$$

$$\frac{\partial^2 f}{\partial x\partial y} = -\sin(x+z^y)\cdot z^y\cdot\ln z\cdot\frac{\partial(x+z^y)}{\partial x} = \underline{-\sin(x+z^y)\cdot z^y\cdot\ln z}$$

$$\frac{\partial^2 f}{\partial y^2} = -\sin(x+z^y)\cdot z^y\cdot\ln z\cdot z^y\cdot\ln z + \cos(x+z^y)\cdot\ln z\cdot z^y\cdot\ln z = \underline{z^y\cdot(\ln z)^2\cdot\left[\cos(x+z^y) - z^y\cdot\sin(x+z^y)\right]}$$

$$\frac{\partial^2 f}{\partial z\partial y} = -\sin(x+z^y)\cdot y z^{y-1}\cdot z^y\cdot\ln z + \cos(x+z^y)\cdot y\cdot z^{y-1}\cdot\ln z + \cos(x+z^y)\cdot z^y\cdot\frac{1}{z}$$

$$= \underline{z^{y-1}\cdot\left[-\sin(x+z^y)\cdot y z^y\cdot\ln z + \cos(x+z^y)\cdot y\cdot\ln z + \cos(x+z^y)\right] = \frac{\partial^2 f}{\partial y\partial z}}$$

$$\frac{\partial^2 f}{\partial z^2} = -\sin(x+z^y)\cdot y\cdot z^{y-1}\cdot y\cdot z^{y-1} + \cos(x+z^y)\cdot y\cdot(y-1)\cdot z^{y-2}$$

$$= \underline{y z^{y-1}\cdot\left[-\sin(x+z^y)\cdot y\cdot z^{y-1} + \cos(x+z^y)\cdot(y-1)\cdot z^{-1}\right]}$$

#20. $f(x,y,z) = xe^y + ye^z + ze^x$

$$\frac{\partial f}{\partial x} = e^y + ze^x \qquad \frac{\partial f}{\partial y} = xe^y + e^z \qquad \frac{\partial f}{\partial z} = ye^z + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \underline{ze^x} \qquad \frac{\partial^2 f}{\partial y \partial x} = \underline{e^y} \qquad \frac{\partial^2 f}{\partial z \partial x} = \underline{e^x}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \underline{e^y} \qquad \frac{\partial^2 f}{\partial y^2} = \underline{xe^y} \qquad \frac{\partial^2 f}{\partial z \partial y} = \underline{e^z}$$

$$\frac{\partial^2 f}{\partial x \partial z} = \underline{e^x} \qquad \frac{\partial^2 f}{\partial y \partial z} = \underline{e^z} \qquad \frac{\partial^2 f}{\partial z^2} = \underline{ye^z}$$

#23.

(a) $\frac{\partial f}{\partial x} = x+y \qquad \frac{\partial f}{\partial y} = y-x$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = 1 \neq \frac{\partial^2 f}{\partial x \partial y} = -1 \Rightarrow \boxed{\underline{\text{No}}}$$

(b) $\frac{\partial f}{\partial x} = xy \qquad \frac{\partial f}{\partial y} = xy$

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = x \neq \frac{\partial^2 f}{\partial x \partial y} = y \text{, for } x \neq y.$$

$$\Rightarrow \boxed{\underline{\text{No}}}$$

#26.

(a) $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$, Find $\lim_{(x,y)\to(0,0)} f(x,y)=$ ______

<pf>

① as $(x,y)\to(0,0)$ along the x-axis, that is, $y=0$

then $f(x,0)=\dfrac{x^2}{x^2}=1 \longrightarrow 1$, as $(x,y)\to(0,0)$

② as $(x,y)\to(0,0)$ along the line $y=x$,

then $f(x,x)=\dfrac{0}{2x^2}=0 \longrightarrow 0$, as $(x,y)\to(0,0)$

since ~~these~~ limits are not equal, so the limit does not exist.

(b) $f(x,y)=\dfrac{y^2}{x^2+y^2}$, Find $\lim_{(x,y)\to(0,0)} f(x,y)=$ ______

<pf>

① as $(x,y)\to(0,0)$ along the x-axis, that is, $y=0$

then $f(x,0)=\dfrac{0}{2x^2}=0 \longrightarrow 0$, as $(x,y)\to(0,0)$

② as $(x,y)\to(0,0)$ along the line $x=y$,

then $f(x,x)=\dfrac{x^2}{2x^2}=\dfrac{1}{2} \to \dfrac{1}{2}$, as $(x,y)\to(0,0)$

since these limites are not equal, so the limit does not exist.

7\. $f(x,y) = \dfrac{xy}{x^2+y^2}$, Find $\lim\limits_{(x,y)\to(0,0)} f(x,y) =$ ______

(a) $(x,y) \to (0,0)$ along the x-axis, that is, $y=0$.

then $f(x,0) = \dfrac{0}{x^2} = 0 \to 0$, as $(x,y)\to(0,0)$

(b) $(x,y)\to(0,0)$ along the y-axis, that is, $x=0$.

then $f(0,y) = \dfrac{0}{y^2} = 0 \to 0$, as $(x,y)\to(0,0)$

(c) $(x,y)\to(0,0)$ along the line $y=mx$,

then $f(x,mx) = \dfrac{mx^2}{x^2+m^2x^2} = \dfrac{m^2}{1+m^2} \longrightarrow \dfrac{m^2}{1+m^2}$, as $(x,y)\to(0,0)$

(d) $(x,y)\to(0,0)$ along the spiral $r=\theta$, $\theta\geq 0$. Let $x=r\cos\theta$ $y=r\sin\theta$.

$f(r,\theta) = \dfrac{r\cos\theta\, r\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \dfrac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta$

then $f(\theta,\theta) = \cos\theta\cdot\sin\theta \longrightarrow 0$, as $(x,y)\to(0,0)$, that is, $r\to0^+$, that is, $\theta\to0^+$

(e) $(x,y)\to(0,0)$ along differentiable $y=f(x)$ with $f(0)=0$

$f(x,f(x)) = \dfrac{x\cdot f(x)}{x^2+(f(x))^2}$, as $(x,y)\to(0,0)$, that is, $x\to0$

$\dfrac{xf(x)}{x^2+(f(x))^2} = \dfrac{\frac{f(x)}{x}}{1+\frac{(f(x))^2}{x^2}} = \dfrac{\frac{f(x)}{x}}{1+\left(\frac{f(x)}{x}\right)^2}$

Now, $\lim_{x\to 0} \frac{f(x)}{x} = \left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{f'(x)}{1} = f'(0)$

Thus, $\lim_{x\to 0} \frac{x\cdot f(x)}{x^2+(f(x))^2} = \lim_{x\to 0} \frac{\frac{f(x)}{x}}{1+\left(\frac{f(x)}{x}\right)^2} = \frac{f'(0)}{1+(f'(0))^2}$

(f) $(x,y)\to(0,0)$ along $r=\sin 3\theta$; $\frac{\pi}{6}<\theta<\frac{1}{3}\pi \Rightarrow \frac{\pi}{2}<3\theta<\pi$

$\Rightarrow r^2=(\sin 3\theta)^2 = x^2+y^2 \Rightarrow x=\cos\theta \sin 3\theta$ and $y=\sin\theta\cdot\sin 3\theta$

as $(x,y)\to(0,0)$, that is, $x\to 0$ and $y\to 0$, that is, $\sin 3\theta \to 0$, that is, $3\theta\to\pi$, $3\theta<\pi$

so, $\theta\to\left(\frac{\pi}{3}\right)^-$

$f(r,\theta)= \frac{\cos\theta \sin 3\theta\cdot \sin\theta \sin 3\theta}{(\cos\theta \sin 3\theta)^2+(\sin\theta \sin 3\theta)^2} = \cos\theta\cdot\sin\theta$

then $\lim_{\theta\to(\frac{\pi}{3})^-} f(r,\theta) = \lim_{\theta\to(\frac{\pi}{3})^-} \sin\theta\cdot\cos\theta = \frac{\sqrt{3}}{4}$

(g) $(x,y)\to(0,0)$ along $r(t)=\frac{1}{t}\vec{i}+\frac{\sin t}{t}\vec{j}$, $t>0$, $\Rightarrow x=\frac{1}{t}$ $y=\frac{\sin t}{t}$

$f(x,y)=\frac{\frac{1}{t}\cdot\frac{\sin t}{t}}{\left(\frac{1}{t}\right)^2+\left(\frac{\sin t}{t}\right)^2} = \frac{\sin t}{1+\sin^2 t}$

as $(x,y)\to(0,0)$, that is, $x\to 0$, $y\to 0$, that is, $t\to\infty$

then $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{\sin t}{1+\sin^2 t}$ does not exist ($\because \lim_{t\to\infty} \sin t$ does not exist)

§ 16-1

※ 6. $f(x,y)=\ln(x^2+y^2)$

$$\frac{\partial f}{\partial x}=\frac{2x}{x^2+y^2} \qquad \frac{\partial f}{\partial y}=\frac{2y}{x^2+y^2} \Rightarrow \nabla f=\frac{2x}{x^2+y^2}\vec{i}+\frac{2y}{x^2+y^2}\vec{j}$$

※ 14. $f(x,y,z)=e^{\frac{yz^2}{x^3}}$

$$\frac{\partial f}{\partial x}=e^{\frac{yz^2}{x^3}}\cdot\frac{0-yz^2\cdot 3x^2}{x^6}=e^{\frac{yz^2}{x^3}}\cdot\frac{-3yz^2}{x^4}$$

$$\frac{\partial f}{\partial y}=e^{\frac{yz^2}{x^3}}\cdot\frac{z^2}{x^3} \qquad \frac{\partial f}{\partial z}=e^{\frac{yz^2}{x^3}}\cdot\frac{2yz}{x^3}$$

$$\Rightarrow \nabla f=e^{\frac{yz^2}{x^3}}\left(\frac{-3yz^2}{x^4}\vec{i}+\frac{z^2}{x^3}\vec{j}+\frac{2yz}{x^3}\vec{k}\right)$$

※ 18. $f(x,y)=2x(x-y)^{-1}=\frac{2x}{x-y}$, $P(3,1)$,

$$\frac{\partial f}{\partial x}=\frac{2(x-y)-2x\cdot 1}{(x-y)^2}=\frac{-2y}{(x-y)^2} \qquad \frac{\partial f}{\partial y}=\frac{0-2x\cdot(-1)}{(x-y)^2}=\frac{2x}{(x-y)^2}$$

$$\nabla f=\frac{-2y}{(x-y)^2}\vec{i}+\frac{2x}{(x-y)^2}\vec{j}$$

$$\Rightarrow \nabla f(3,1)=\frac{-1}{2}\vec{i}+\frac{3}{2}\vec{j}$$

※22.

$f(x,y) = xy\,e^{-(x^2+y^2)}$ $\quad P(1,-1)$

$$\frac{\partial f}{\partial x} = y e^{-x^2-y^2} + xy\cdot e^{-x^2-y^2}\cdot(-2x) = e^{-x^2-y^2}\cdot(y-2x^2y)$$

$$\frac{\partial f}{\partial y} = x\cdot e^{-x^2-y^2} + xy\cdot e^{-x^2-y^2}\cdot(-2y) = e^{-x^2-y^2}\cdot(x-2xy^2)$$

$$\nabla f = e^{-x^2-y^2}\cdot\left((y-2x^2y)\vec{i} + (x-2xy^2)\vec{j}\right)$$

$$\Rightarrow \nabla f(1,-1) = e^{-2}\cdot(\vec{i}-\vec{j})$$

※30.

$f(x,y) = \frac{1}{2}x^2 + 2xy + y^2$

$\vec{x} = (x,y)$ $\quad \vec{h} = (h_1, h_2)$

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = f(x+h_1, y+h_2) - f(x,y)$$

$$= \left[\frac{1}{2}(x+h_1)^2 + 2(x+h_1)(y+h_2) + (y+h_2)^2\right] - \left[\frac{1}{2}x^2 + 2xy + y^2\right]$$

$$= xh_1 + \frac{1}{2}h_1^2 + 2xh_2 + 2yh_1 + 2h_1h_2 + 2yh_2 + h_2^2$$

$$= xh_1 + 2xh_2 + 2yh_1 + 2yh_2 + \frac{1}{2}h_1^2 + 2h_1h_2 + h_2^2$$

$$= (x+2y)h_1 + (2x+2y)h_2 + o(\vec{h})$$

$$= \nabla f(\vec{x})\cdot\vec{h} + o(\vec{h})$$

where $\nabla f(\vec{x}) = \nabla f(x,y) = (x+2y,\ 2x+2y)$

$\vec{h} = (h_1, h_2)$

$o(\vec{h}) = o(h_1, h_2) = \frac{1}{2}h_1^2 + 2h_1h_2 + h_2^2$

39. $f(x,y)=1+x^2+y^2$

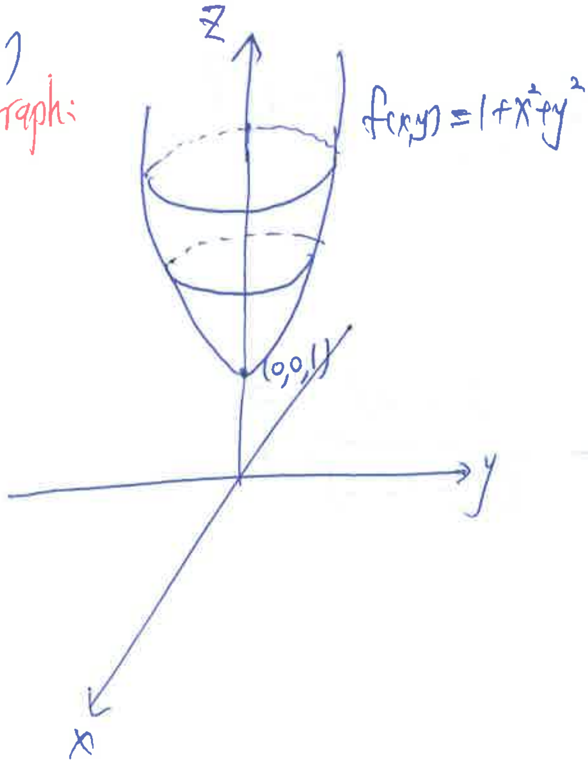
(a)

$\frac{\partial f}{\partial x}=2x$, $\frac{\partial f}{\partial y}=2y \Rightarrow \nabla f(x,y)=2x\vec{i}+2y\vec{j}$

Let $\nabla f(x,y)=(2x,2y)=(0,0) \Rightarrow \begin{matrix} 2x=0 \\ 2y=0 \end{matrix} \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix}$

So, $\nabla f(\vec{x})=\vec{0}$, then $\vec{x}=(0,0)$

(b)
graph:



(c)

$f(0,0)=1$

since $\nabla f(0,0)=\vec{0}$

so, f has an absolute minimum at $(0,0)$

§16-2

※6.

$f(x,y) = \dfrac{x+y}{cx+dy}$ at $P(1,1)$ in the direction of $c\vec{i} - d\vec{j}$

$$\frac{\partial f}{\partial x} = \frac{(cx+dy)-(x+y)\cdot c}{(cx+dy)^2} = \frac{(d-c)y}{(cx+dy)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(cx+dy)-(x+y)\cdot d}{(cx+dy)^2} = \frac{(c-d)x}{(cx+dy)^2}$$

$$\nabla f(x,y) = \frac{(d-c)y}{(cx+dy)^2}\vec{i} + \frac{(c-d)x}{(cx+dy)^2}\vec{j} \qquad \vec{u} = \frac{1}{\sqrt{c^2+d^2}}\cdot(c\vec{i} - d\vec{j})$$

$$\nabla f(1,1) = \frac{d-c}{(c+d)^2}\vec{i} + \frac{c-d}{(c+d)^2}\vec{j}$$

$$\Rightarrow f'_{\vec{u}}(1,1) = \nabla f(1,1)\cdot\vec{u} = \frac{c(d-c)}{(c+d)^2\cdot\sqrt{c^2+d^2}} - \frac{d(c-d)}{\sqrt{c^2+d^2}\cdot(c+d)^2}$$

$$= \frac{d-c}{(c+d)\cdot\sqrt{c^2+d^2}}$$

10. $f(x,y,z) = x^2y + y^2z + z^2x$ at $P(1,0,1)$ in the direction $3\vec{j} - \vec{k}$.

$$\frac{\partial f}{\partial x} = 2xy + z^2 \qquad \frac{\partial f}{\partial y} = x^2 + 2yz \qquad \frac{\partial f}{\partial z} = y^2 + 2zx$$

$$\nabla f(x,y,z) = (2xy + z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 2zx)\vec{k}$$

$$\nabla f(1,0,1) = \vec{i} + \vec{j} + 2\vec{k}$$

$$\vec{u} = \frac{3}{\sqrt{10}}\vec{j} - \frac{1}{\sqrt{10}}\vec{k}$$

$$\Rightarrow f_{\vec{u}}(1,0,1) = \nabla f(1,0,1) \cdot \vec{u}$$

$$= (\vec{i} + \vec{j} + 2\vec{k}) \cdot \left(\frac{3}{\sqrt{10}}\vec{j} - \frac{1}{\sqrt{10}}\vec{k}\right)$$

$$= \frac{3}{\sqrt{10}} - \frac{2}{\sqrt{10}}$$

$$= \frac{1}{\sqrt{10}}$$

16. $f(x,y) = (x-1)\cdot y^2 \cdot e^{xy}$ at $(0,1)$ toward the point $(-1,3)$.

$$\frac{\partial f}{\partial x} = y^2 \cdot e^{xy} + (x-1)\cdot y^2 \cdot e^{xy}\cdot y = e^{xy}\left[y^2 + xy^3 - y^3\right]$$

$$\frac{\partial f}{\partial y} = 2y(x-1)\cdot e^{xy} + (x-1)\cdot y^2 \cdot e^{xy}\cdot x = e^{xy}\left[2xy - 2y + x^2y^2 - xy^2\right]$$

$$\nabla f(x,y) = e^{xy}\cdot\left[y^2 + xy^3 - y^3\right]\vec{i} + e^{xy}\left[2xy - 2y + x^2y^2 - xy^2\right]\vec{j}$$

$\nabla f(0,1) = -2\vec{j}$ and in the direction $(-1,3)-(0,1) = (-1, 2)$

$$\vec{u} = \frac{1}{\sqrt{5}}(-1,2) = \left(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{-1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}$$

$$\Rightarrow f_{\vec{u}}(0,1) = (-2\vec{j})\cdot\left(\frac{-1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{j}\right) = \frac{-4}{\sqrt{5}}$$

22.

$f(x,y,z) = e^{x} \cdot \cos(\pi y z)$ at $(0, 1, \frac{1}{2})$ in the directions parallel to the line in which the planes $x+y-z=5$ and $4x-y-z=2$ intersect.

$$\frac{\partial f}{\partial x} = e^{x} \cdot \cos(\pi y z) \qquad \frac{\partial f}{\partial y} = e^{x} \cdot (-\sin(\pi y z)) \cdot \pi z = -e^{x} \cdot \pi z \cdot \sin(\pi y z)$$

$$\frac{\partial f}{\partial z} = e^{x} \cdot (-\sin(\pi y z)) \cdot \pi y = -e^{x} \cdot \pi y \cdot \sin(\pi y z)$$

$$\nabla f(x,y,z) = e^{x}\cos(\pi y z)\,\vec{i} + [-e^{x}\pi z \cdot \sin(\pi y z)]\,\vec{j} + [-e^{x} \cdot \pi y \cdot \sin(\pi y z)]\,\vec{k}$$

$$\nabla f(0,1,\tfrac{1}{2}) = -\frac{\pi}{2}\vec{j} - \pi \vec{k}$$

$$\vec{v} = \pm(\vec{i} + \vec{j} - \vec{k}) \times (4\vec{i} - \vec{j} - \vec{k})$$

$$= \pm\left(\begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ -1 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} \right) = \pm(-2, -3, -5)$$

$$= \pm(2\vec{i} + 3\vec{j} + 5\vec{k})$$

$$\vec{u} = \pm\frac{1}{\sqrt{38}}(2\vec{i} + 3\vec{j} + 5\vec{k}) = \pm\left(\frac{2}{\sqrt{38}}\vec{i} + \frac{3}{\sqrt{38}}\vec{j} + \frac{5}{\sqrt{38}}\vec{k}\right)$$

$$\Rightarrow f_{\vec{u}}(0,1,\tfrac{1}{2}) = \left(\frac{-\pi}{2}\vec{j} - \pi\vec{k}\right) \cdot \left(\pm\left(\frac{2}{\sqrt{38}}\vec{i} + \frac{3}{\sqrt{38}}\vec{j} + \frac{5}{\sqrt{38}}\vec{k}\right)\right)$$

$$= \mp\left(\frac{3\pi}{2\sqrt{38}} + \frac{5\pi}{\sqrt{38}}\right)$$

$$= \mp\frac{13\pi}{2\sqrt{38}}$$

24.

$f(x,y) = x + \sin(x+2y)$ at $P(0,0)$.

$\frac{\partial f}{\partial x} = 1 + \cos(x+2y)$ $\qquad$ $\frac{\partial f}{\partial y} = 2\cos(x+2y)$

$\nabla f(x,y) = (1+\cos(x+2y))\vec{i} + 2\cos(x+2y)\vec{j}$

$\nabla f(0,0) = 2\vec{i} + 2\vec{j}$ $\qquad$ $\|\nabla f(0,0)\| = \sqrt{8} = 2\sqrt{2}$

Fastest increase in direction $\vec{u} = \frac{1}{2\sqrt{2}}(2\vec{i}+2\vec{j}) = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$, rate of change $= 2\sqrt{2}$

Fastest decrease in direction $\vec{v} = \frac{-1}{2\sqrt{2}}(2\vec{i}+2\vec{j}) = \frac{-1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$, rate of change $= -2\sqrt{2}$

28.

$\nabla f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\vec{i} + \frac{\partial f}{\partial y}(x_0,y_0)\vec{j} \neq \vec{0}$

$\vec{c} = \frac{\partial f}{\partial y}(x_0,y_0)\vec{i} - \frac{\partial f}{\partial x}(x_0,y_0)\vec{j}$

$\Rightarrow f_{\vec{c}}(x_0,y_0) = \nabla f(x_0,y_0) \cdot \vec{c}$

$= \frac{\partial f}{\partial x}(x_0,y_0) \cdot \frac{\partial f}{\partial y}(x_0,y_0) - \frac{\partial f}{\partial y}(x_0,y_0)\frac{\partial f}{\partial x}(x_0,y_0)$

$= 0$

Thus, the vector $\vec{c}$ is perpendicular to $\nabla f(x_0,y_0)$ and points along the level curve of f at (x_0,y_0).

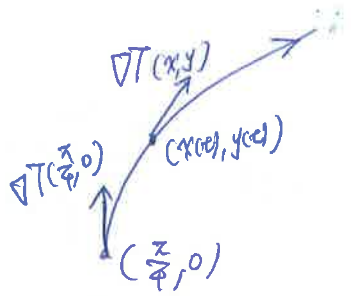
37.

$T(x,y) = \sqrt{2}e^{-y} \cdot \cos x$ at $P(\frac{\pi}{4}, 0)$

$$\frac{\partial T}{\partial x} = \sqrt{2}e^{-y} \cdot (-\sin x) = -\sqrt{2}e^{-y} \cdot \sin x$$

$$\frac{\partial T}{\partial y} = \sqrt{2}e^{-y} \cdot (-1) \cdot \cos x = -\sqrt{2}e^{-y} \cdot \cos x$$

(curve)
$C: r(t) = x(t)\vec{i} + y(t)\vec{j}$



$$\nabla T(x,y) = -\sqrt{2}e^{-y}\sin x\, \vec{i} - \sqrt{2}e^{-y}\cos x\, \vec{j}$$

$$\nabla T(\frac{\pi}{4}, 0) = -\vec{i} - \vec{j}$$

Let curve $C: r(t) = x(t)\vec{i} + y(t)\vec{j}$ satisfies $r'(t) = \nabla T(x,y)$

$r'(t) = \nabla T(x,y) \Rightarrow x'(t)\vec{i} + y'(t)\vec{j} = -\sqrt{2}e^{-y}\sin x\,\vec{i} - \sqrt{2}e^{-y}\cos x\,\vec{j}$

$\Rightarrow x'(t) = -\sqrt{2}e^{-y}\sin x$, $y'(t) = -\sqrt{2}e^{-y}\cos x$

$\Rightarrow \frac{x'(t)}{y'(t)} = \tan x \Rightarrow \frac{dx}{dy} = \tan x \Rightarrow \frac{dy}{dx} = \cot x \Rightarrow dy = \cot x\, dx$

$\Rightarrow y = \int \cot x\, dx = \ln|\sin x| + C$

$\Rightarrow 0 = \ln|\sin\frac{\pi}{4}| + C \Rightarrow C = \ln\sqrt{2}$

$\therefore y = \ln|\sin x| + \ln\sqrt{2} = \ln|\sqrt{2}\sin x|$. ← we want to find this curve. !!

As $\nabla T(\frac{\pi}{4}, 0) = -\vec{i} - \vec{j}$, the curve $y = \ln|\sqrt{2}\sin x|$ is followed in the direction of decreasing x.

40. $\vec{F}(x,y,z) = -\dfrac{GMm}{r^3}\vec{r}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r = \|\vec{r}\|$.

G is the universal gravitational constant.

Show that $\vec{F}$ is the gradient of the function $f(x,y,z) = \dfrac{GMm}{r}$.

<pf>

$$f(x,y,z) = \frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial f}{\partial x} = \frac{-GMm\cdot 2x}{2(x^2+y^2+z^2)^{3/2}} = \frac{-GMmx}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial f}{\partial y} = \frac{-GMm\cdot 2y}{2(x^2+y^2+z^2)^{3/2}} = \frac{-GMmy}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial f}{\partial z} = \frac{-GMm\cdot 2z}{2(x^2+y^2+z^2)^{3/2}} = \frac{-GMmz}{(x^2+y^2+z^2)^{3/2}}$$

$$\nabla f(x,y,z) = \frac{-GMm}{(x^2+y^2+z^2)^{3/2}}(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= \frac{-GMm}{r^3}\cdot\vec{r}$$, where $r^3 = \|\vec{r}\|^3 = (x^2+y^2+z^2)^{3/2}$.

$\blacksquare$