Hw 1 :

section 11-2 ※ 16, 18, 26, 36, 40, 46.

16.

$a_n = \sqrt{n^2+1}$ $\quad \because n^2+1 > n^2$ and $a_n > 0$, for each $n \in \mathbb{N}$

$\Rightarrow \sqrt{n^2+1} > \sqrt{n^2} = n$

$\Rightarrow a_n > n$

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{(n+1)^2+1}}{\sqrt{n^2+1}} = \sqrt{\frac{n^2+2n+2}{n^2+1}} = \sqrt{1+\frac{2n+1}{n^2+1}} > \sqrt{1} = 1$$

$\Rightarrow a_{n+1} > a_n$

$\Rightarrow$ increasing, that is, $a_1 < a_2 < a_3 < \cdots\cdots$

thus, bounded below by $a_1 = \sqrt{1^2+1} = \sqrt{2}$

$\because a_n > n$, $\therefore$ not bounded above.

18.

$$a_n = \frac{2^n}{4^n+1} = \frac{1}{\frac{4^n}{2^n}+\frac{1}{2^n}} = \frac{1}{2^n+\frac{1}{2^n}} \geq 0$$

$a_1 = \frac{2}{5}$, $a_2 = \frac{4}{17}$, $a_3 = \frac{8}{65}$, $a_4 = \frac{16}{257}$, $\cdots\cdots$

$\Rightarrow a_n$ is decreasing, that is, $a_1 > a_2 > a_3 > a_4 > \cdots\cdots$

$\Rightarrow$ bounded above by $a_1 = \frac{2}{5}$

bounded below by 0

Let $$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{4^{n+1}+1}}{\frac{2^n}{4^n+1}} = \frac{2^{n+1}}{2^n} \cdot \frac{4^n+1}{4^{n+1}+1}$$

$$\leq 2 \cdot \frac{4^n+1}{4^{n+1}}$$

$$= 2 \cdot \left(\frac{4^n}{4^{n+1}} + \frac{1}{4^{n+1}}\right)$$

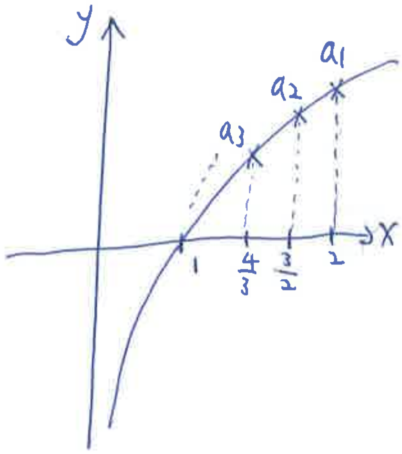
$$< 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = 1$$

$\therefore a_{n+1} < a_n$ (decreasing).

26.

$a_n = \ln\left(\frac{n+1}{n}\right)$

$\because \frac{n+1}{n}$ is decreasing to 1 and $\ln x$ is increasing on $x>0$



$\Rightarrow$ a_n is decreasing, that is, $a_1 > a_2 > a_3 > \cdots\cdots$

$\Rightarrow$ bounded above by $a_1 = \ln 2$

bounded below by 0

36.

$a_n = \cos(n\pi)$

$a_1 = \cos\pi = -1$

$a_2 = \cos 2\pi = 1$

$a_3 = \cos 3\pi = \cos\pi = -1$

$a_4 = \cos 4\pi = \cos 2\pi = 1$

$\vdots$

$\Rightarrow$ not monotonic

$\Rightarrow$ bounded above by 1

bounded below by -1

40. $a_n = \dfrac{1-\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^n} = \dfrac{\left(1-\left(\frac{1}{2}\right)^n\right)\cdot 2^n}{\left(\frac{1}{2}\right)^n \cdot 2^n} = 2^n - 1$

$a_1 = 1$

$a_2 = 3$

$a_3 = 7$

$a_4 = 15$

$\vdots$

$\Rightarrow$ a_n is increasing, that is, $a_1 < a_2 < a_3 < \cdots\cdots$

$\Rightarrow$ bounded below by $a_1 = 1$

not bounded above.

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46.

$a_1 = 1$, $a_{n+1} = a_n + 3n(n+1) + 1$

the first six terms: $a_1 = 1 = 1^3$, $a_4 = a_3 + 36 + 1 = 64 = 4^3$

$a_2 = a_1 + 6 + 1 = 8 = 2^3$, $a_5 = a_4 + 60 + 1 = 125 = 5^3$

$a_3 = a_2 + 18 + 1 = 27 = 3^3$, $a_6 = a_5 + 90 + 1 = 216 = 6^3$

$\Rightarrow a_n = n^3$

P364. Project 7.4

(7.4.11) $\left(1+\frac{1}{n}\right)^n \leq e \leq \left(1+\frac{1}{n}\right)^{n+1}$, $n = 1, 2, 3, 4, \cdots\cdots$ (we want to prove it)

Define:

$\ln x = \int_1^x \frac{1}{t}\, dt$, $x > 0$

Let $\ln x = 1$, we have only number $x = e$ for which $\ln e = 1$.

Step 1:

$\frac{1}{n+1} \leq \ln\left(1+\frac{1}{n}\right) \leq \frac{1}{n}$, $n = 1, 2, 3, \cdots\cdots$

Hint: $t \in [1, 1+\frac{1}{n}] \Rightarrow \frac{1}{1+\frac{1}{n}} \leq \frac{1}{t} \leq 1$, $n = 1, 2, 3, \cdots\cdots$

Step 2:

$\left(1+\frac{1}{n}\right)^n \leq e \leq \left(1+\frac{1}{n}\right)^{n+1}$

by applying the exponential function to each entry in the inequality derived in Step 1.

<pf of step 1>

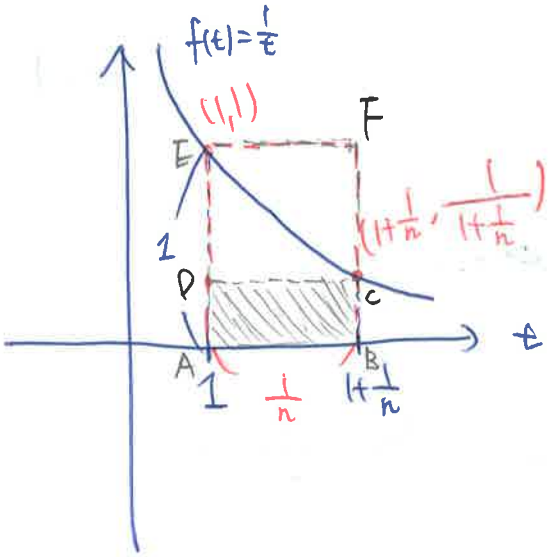
$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x>0$$

① $\because f(t)=\frac{1}{t}$ is continuous on $t>1$ $\therefore$ the definite integral $\ln x=\int_1^x \frac{1}{t}\,dt$ is defined.

②

by F.T.C. $\frac{d\ln x}{dx}=\frac{1}{x}>0$, as $x>0 \Rightarrow \ln x$ is increasing. and $\ln 1=\int_1^1 \frac{1}{t}\,dt=0$

③ $\ln(1+\frac{1}{n}) = \int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt$, by the graph of $f(t)=\frac{1}{t}$:



$\Rightarrow$ Area of ABCD $\leq \int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt \leq$ Area of ABFE

$\Rightarrow \frac{1}{n}\cdot\frac{1}{1+\frac{1}{n}} \leq \ln(1+\frac{1}{n}) \leq \frac{1}{n}\cdot 1$

$\Rightarrow \frac{1}{n+1} \leq \ln(1+\frac{1}{n}) \leq \frac{1}{n}$, $n=1,2,3,\cdots\cdots$

<pf of step 2>

④ $\because \ln x$ is increasing, that is, $x<y \Rightarrow \ln x \leq \ln y$

by step 1 $\Rightarrow \frac{1}{n+1} \leq \ln(1+\frac{1}{n}) \leq \frac{1}{n}$

$\Rightarrow (n+1)\cdot\frac{1}{n+1} \leq (n+1)\cdot\ln(1+\frac{1}{n})$ and $n\cdot\ln(1+\frac{1}{n}) \leq n\cdot\frac{1}{n}$

$\Rightarrow 1 \leq \ln(1+\frac{1}{n})^{n+1}$ $\qquad \ln(1+\frac{1}{n})^n \leq 1$

$\Rightarrow \ln e \leq \ln(1+\frac{1}{n})^{n+1}$ $\qquad \ln(1+\frac{1}{n})^n \leq \ln e$

$\Rightarrow e \leq (1+\frac{1}{n})^{n+1}$ $\qquad (1+\frac{1}{n})^n \leq e$

thus, $(1+\frac{1}{n})^n \leq e \leq (1+\frac{1}{n})^{n+1}$ ∎

<pf of sep 17> $\boxed{\text{other proof}}$

$\because\ t\in[1, 1+\frac{1}{n}] \Rightarrow 1 \le t \le 1+\frac{1}{n}$

$$\Rightarrow \frac{1}{1+\frac{1}{n}} \le \frac{1}{t} \le 1$$

$$\Rightarrow \int_1^{1+\frac{1}{n}} \frac{1}{1+\frac{1}{n}}\, dt \le \int_1^{1+\frac{1}{n}} \frac{1}{t}\, dt \le \int_1^{1+\frac{1}{n}} 1\, dt$$

$$\Rightarrow \frac{1}{1+\frac{1}{n}} \cdot t \Big|_{t=1}^{t=1+\frac{1}{n}} \le \ln\left(1+\frac{1}{n}\right) \le t \Big|_{t=1}^{t=1+\frac{1}{n}}$$

$$\Rightarrow 1 - \frac{1}{1+\frac{1}{n}} \le \ln\left(1+\frac{1}{n}\right) \le \left(1+\frac{1}{n}\right) - 1$$

$$\Rightarrow \frac{\frac{1}{n}}{1+\frac{1}{n}} \le \ln\left(1+\frac{1}{n}\right) \le \frac{1}{n}$$

$$\Rightarrow \frac{1}{n+1} \le \ln\left(1+\frac{1}{n}\right) \le \frac{1}{n}, \quad n=1,2,3,\ldots\ldots$$

$\blacksquare$

§11-2

$\divideontimes$ 46: claim: $a_n = n^3$

$\because\ a_{n+1} = a_n + 3n(n+1) + 1 = a_n + 3n^2 + 3n + 1$

$$\left.\begin{aligned}
a_{n+1} &= \cancel{a_n} + 3n^2 + 3n + 1 \\
\cancel{a_n} &= \cancel{a_{n-1}} + 3(n-1)^2 + 3(n-1) + 1 \\
\cancel{a_{n-1}} &= \cancel{a_{n-2}} + 3(n-2)^2 + 3(n-2) + 1 \\
&\ \vdots \\
\cancel{a_3} &= \cancel{a_2} + 3\cdot 2^2 + 3\cdot 2 + 1 \\
+)\ \cancel{a_2} &= a_1 + 3\cdot 1^2 + 3\cdot 1 + 1
\end{aligned}\right\}\ n \text{ rows}$$

$\Rightarrow a_{n+1} + \cancel{a_n} + \cancel{\cdots} + \cancel{a_2} = (\cancel{a_n} + \cancel{a_{n-1}} \cancel{+} \cdots + \cancel{a_2} + a_1) + 3\cdot[n^2 + (n-1)^2 + \cdots + 2^2 + 1^2]$

$+ 3\cdot[n + (n-1) + \cdots + 2 + 1]$

$+ n\cdot 1$

$\Rightarrow a_{n+1} = a_1 + 3\cdot\dfrac{n(n+1)(2n+1)}{6} + 3\cdot\dfrac{n(n+1)}{2} + n,\quad a_1 = 1,$

$\Rightarrow a_{n+1} = \dfrac{n(n+1)(2n+1)}{2} + \dfrac{3n(n+1)}{2} + n + 1$

$= \dfrac{(n+1)n}{\cancel{2}}\cdot(\underbrace{2n+1 \cancel{+} 3}_{=\cancel{2}n+4})^{(n+2)} + n + 1 = n(n+1)(n+2) + (n+1)$

$= (n+1)[\underbrace{n^2 + 2n + 1}_{=(n+1)^2}] = (n+1)^3$

$\Rightarrow a_n = n^3.$