V

HW 2: §11-3 * 10, 20, 30, 32

§11-4 * 18, 26, 28, 32, 36, 44, 47

§11-5 * 6, 22, 26, 28, 46, 49

§11-3

*10. $a_n = \frac{n^2}{n+1} \geq \frac{n}{2} \Rightarrow \lim_{n\to\infty} \frac{n^2}{n+1} \geq \lim_{n\to\infty} \frac{n}{2}$ does not exist.!!

$\because n \geq 1 \Rightarrow n+n \geq 1+n \Rightarrow \frac{1}{n+1} \geq \frac{1}{2n}$

$\Rightarrow \frac{n^2}{n+1} \geq \frac{n^2}{2n} = \frac{n}{2}$ $\therefore$ divergent

*20. $a_n = \frac{n^4-1}{n^4+n-6} = \frac{\frac{n^4}{n^4} - \frac{1}{n^4}}{\frac{n^4}{n^4} + \frac{n}{n^4} - \frac{6}{n^4}} = \frac{1-\frac{1}{n^4}}{1+\frac{1}{n^3}-\frac{6}{n^4}} \xrightarrow{\text{as } n\to\infty} \frac{1-0}{1+0-0} = 1$

$\because \lim_{n\to\infty} \frac{1}{n^4} = 0$ $\lim_{n\to\infty} \frac{1}{n^3} = 0$ $\therefore \lim_{n\to\infty} a_n = 1$

*30. $a_n = (1+\frac{1}{n})^{\frac{n}{2}} = ((1+\frac{1}{n})^n)^{\frac{1}{2}}$

$\because f(x) = x^{\frac{1}{2}}$ is continuous on $x \geq 0$, $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} ((1+\frac{1}{n})^n)^{\frac{1}{2}} = \lim_{n\to\infty} f((1+\frac{1}{n})^n) = f(\lim_{n\to\infty} (1+\frac{1}{n})^n) = f(e) = e^{\frac{1}{2}} = \sqrt{e}$

*32. $a_n = 2\ln(3n) - \ln(n^2+1) = \ln(3n)^2 - \ln(n^2+1) = \ln(\frac{9n^2}{n^2+1}) = \ln(\frac{9}{1+\frac{1}{n^2}}) \to \ln 9$, as $n\to\infty$

$\because \lim_{n\to\infty} \frac{1}{n^2} = 0$ $\therefore \lim_{n\to\infty} (2\ln(3n) - \ln(n^2+1)) = \ln 9$

§11-4:

※18.
$$\int_0^n e^{-nx}dx = \int_0^{n^2} e^{-y}\cdot\frac{1}{n}dy = \frac{1}{n}\cdot\int_0^{n^2} e^{-y}dy = \frac{1}{n}\cdot\left(-e^{-y}\Big|_{y=0}^{y=n^2}\right)$$

$y=nx$ $\quad x=n \to y=n^2$

$dy = ndx$ $\quad x=0 \to y=0$

$$= \frac{1}{n}\cdot\left(-e^{-n^2}-(-1)\right)$$
$$= \frac{-e^{-n^2}+1}{n}$$
$$= \frac{-e^{-n^2}}{n}+\frac{1}{n}$$
$$= \frac{-1}{n\cdot e^{n^2}}+\frac{1}{n}$$

$$\therefore \lim_{n\to\infty}\int_0^n e^{-nx}dx$$
$$= \lim_{n\to\infty}\left(\frac{1}{n\cdot e^{n^2}} - \frac{1}{n}\right)$$

$\because \lim_{n\to\infty}\frac{1}{n}=0 \quad \lim_{n\to\infty}\frac{1}{e^{n^2}}=0$

$$= 0-0$$
$$=0$$

※26.
$$a_n = \left(1+\frac{x}{n}\right)^{3n} = \left(\left(1+\frac{x}{n}\right)^n\right)^3$$

(use P552) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$, $\quad x$: real number.

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{3n} = \lim_{n\to\infty}\left(\left(1+\frac{x}{n}\right)^n\right)^3$$
$$= \lim_{n\to\infty}\left(\left(1+\frac{x}{n}\right)^n\cdot\left(1+\frac{x}{n}\right)^n\cdot\left(1+\frac{x}{n}\right)^n\right)$$
$$= e^x \times e^x \times e^x$$
$$= e^{3x}$$

§11-4 13

※ 28. $\int_{\frac{1}{n}}^{1} \frac{1}{\sqrt{x}}dx = \int_{\frac{1}{n}}^{1} x^{-\frac{1}{2}}dx = 2x^{\frac{1}{2}}\Big|_{\frac{1}{n}}^{1} = 2-2\cdot(\frac{1}{n})^{\frac{1}{2}}$

$= 2 - 2\cdot\frac{1}{\sqrt{n}}$

$\lim_{n\to\infty}\int_{\frac{1}{n}}^{1}\frac{1}{\sqrt{x}}dx = \lim_{n\to\infty}(2-2\cdot\frac{1}{\sqrt{n}})$ $\because \lim_{n\to\infty}\frac{1}{\sqrt{n}}=0 \quad \lim_{n\to\infty}2=2$

$= 2-2\times 0$

$= 2$

※32 $\lim_{n\to\infty}\int_0^{\frac{1}{n}}\cos(e^x)\,dx =$ ____

$\because -1\le\cos(e^x)\le 1 \Rightarrow \int_0^{\frac{1}{n}} -1\,dx \le \int_0^{\frac{1}{n}}\cos(e^x)\,dx \le \int_0^{\frac{1}{n}} 1\,dx$

$\Rightarrow \frac{-1}{n} \le \int_0^{\frac{1}{n}}\cos(e^x)\,dx \le \frac{1}{n}$

by squeeze theorem $\lim_{n\to\infty}\frac{-1}{n} = \lim_{n\to\infty}\frac{1}{n} = 0$

$\Rightarrow \lim_{n\to\infty}\int_0^{\frac{1}{n}}\cos(e^x)\,dx = 0$

※36 $\lim_{n\to\infty}(|t|+\frac{x}{n})^n =$ ____ , when $t>0$, $x>0$,

$(t+\frac{x}{n})^n = [t(1+\frac{x}{tn})]^n = t^n\cdot(1+\frac{x}{tn})^n = t^n\cdot(1+\frac{\frac{x}{t}}{n})^n$

case 1: if $0<t<1$, then $\lim_{n\to\infty}t^n=0$ $\lim_{n\to\infty}(1+\frac{\frac{x}{t}}{n})^n = e^{\frac{x}{t}} \Rightarrow \lim_{n\to\infty}(t+\frac{x}{n})^n = 0$

case 2: if $t=1$, $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$

case 3: if $t>1$, $\because t^n\cdot(1+\frac{\frac{x}{t}}{n})^n \ge t^n \Rightarrow \lim_{n\to\infty}(t+\frac{x}{n})^n \ge \lim_{n\to\infty}t^n =$ does not exist

$\therefore$ diverges

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44.

$$\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{(1+n)(2+n)} = \_\_\_\_$$

$\because$ $1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$\therefore \frac{1^2+2^2+\cdots+n^2}{(1+n)(2+n)} = \frac{\frac{n(\cancel{n+1})(2n+1)}{6}}{(\cancel{n+1})(2+n)} = \frac{n(2n+1)}{6(n+2)}$$

$$\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{(1+n)(2+n)} = \lim_{n\to\infty} \frac{n(2n+1)}{6(n+2)} = \lim_{n\to\infty} \frac{2n^2+n}{6n+12}$$

$$= \lim_{n\to\infty} \frac{\frac{2n^2}{n^2}+\frac{n}{n^2}}{\frac{6n}{n^2}+\frac{12}{n^2}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{\frac{6}{n}+\frac{12}{n^2}} = \frac{2}{0}$$ does not exist.

$\therefore$ divergence

47. a sequence $\{a_n\}$, $m_n = \dfrac{a_1+a_2+\cdots+a_n}{n}$

(a) if a_n: increasing sequence, then m_n: increasing sequence

(b) if $a_n \to 0$, then $m_n \to 0$.

<pf> (a)

$$m_{n+1} - m_n = \frac{a_1+a_2+\cdots+a_n+a_{n+1}}{n+1} - \frac{a_1+a_2+\cdots+a_n}{n} = \frac{n\cdot a_{n+1} - (a_1+a_2+\cdots+a_n)}{n(n+1)}$$

$\because$ $a_1+a_2+\cdots+a_n \le n\cdot a_n \Rightarrow m_{n+1} - m_n \geq \dfrac{n\cdot a_{n+1} - n\cdot a_n}{n(n+1)} = \dfrac{a_{n+1}-a_n}{n+1} \geq 0$

($\because$ a_n: increasing)

$\Rightarrow m_{n+1} \geq m_n$, for all n;

$\Rightarrow m_n$: increasing sequence

§ 11-4 : $\quad m_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$ (5

47. (b). if $a_n \to 0$, then $m_n \to 0$

<pf> claim: $\lim_{n\to\infty} m_n = 0$.

$$|m_n| = \frac{|a_1 + a_2 + \cdots + a_n|}{n}$$

$\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $|a_n| < \varepsilon/2$, as $n \geq N_0$

$$|m_n| = \frac{|a_1 + a_2 + \cdots + a_{N_0} + a_{N_0+1} + \cdots + a_n|}{n}$$

$$\leq \frac{|a_1 + a_2 + \cdots + a_{N_0}|}{n} + \frac{\overbrace{|a_{N_0+1}| + |a_{N_0+2}| + \cdots + |a_n|}^{n-N_0 \text{ terms.}}}{n}$$

$$\leq \frac{|a_1 + a_2 + \cdots + a_{N_0}|}{n} + \frac{\varepsilon}{2} \cdot \frac{n-N_0}{n}$$

$\because \lim_{n\to\infty} \dfrac{|a_1 + a_2 + \cdots + a_{N_0}|}{n} = 0$

for this $\varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ s.t. $\dfrac{|a_1 + a_2 + \cdots + a_{N_0}|}{n} < \varepsilon/2$, as $n \geq N_1$

Let $N = \max\{N_0, N_1\} \in \mathbb{N}$ such that as $n \geq N \Rightarrow n \geq N_0$ and $n \geq N_1$

$$\Rightarrow |m_n| \leq \frac{|a_1 + a_2 + \cdots + a_{N_0}|}{n} + \frac{\varepsilon}{2} \cdot \frac{n-N_0}{n} \quad \left(\because \frac{n-N_0}{n} < 1 \Rightarrow \frac{\varepsilon}{2} \cdot \frac{n-N_0}{2} < \frac{\varepsilon}{2}\right)$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon.$$

$\therefore |m_n| < \varepsilon$, as $n \geq N$ $\quad \therefore \lim_{n\to\infty} m_n = 0$ ■

§11-5

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*6.

$$\lim_{x\to a} \frac{x-a}{x^n-a^n} = \_\_\_\_$$

$f(x)=x-a$, $g(x)=x^n-a^n$, $\frac{f(x)}{g(x)} \to \frac{0}{0}$, as $x\to a$,

$$\frac{f'(x)}{g'(x)} = \frac{1}{n\cdot x^{n-1}} \longrightarrow \frac{1}{n\cdot a^{n-1}} \text{, as } x\to a.$$

$$\therefore \lim_{x\to a} \frac{x-a}{x^n-a^n} = \frac{1}{n\cdot a^{n-1}}$$

*22.

$$\lim_{x\to 0} \frac{x\cdot e^{nx}-x}{1-\cos(nx)} = \_\_\_\_$$

$f(x)=x\cdot e^{nx}-x$. $g(x)=1-\cos(nx)$, $\frac{f(x)}{g(x)} \to \frac{0}{0}$, as $x\to 0$,

$$\frac{f'(x)}{g'(x)} = \frac{e^{nx}+n\cdot x\cdot e^{nx}-1}{-(-\sin(nx))\cdot n} = \frac{e^{nx}+n\cdot x\cdot e^{nx}-1}{n\cdot \sin(nx)} \to \frac{0}{0} \text{, as } x\to 0$$

$$\frac{f''(x)}{g''(x)} = \frac{n\cdot e^{nx}+n\cdot e^{nx}+n^2x\cdot e^{nx}}{n^2\cdot \cos(nx)} = \frac{(2n+n^2x)\cdot e^{nx}}{n^2\cdot\cos(nx)} \to \frac{2n}{n^2} = \frac{2}{n} \text{, as } x\to 0$$

$$\therefore \lim_{x\to 0} \frac{x\cdot e^{nx}-x}{1-\cos(nx)} = \frac{2}{n}$$

16.

$$\lim_{x\to0^+}\frac{\sqrt{x}}{\sqrt{x}+\sin\sqrt{x}} = \underline{\qquad}$$

$f(x)=\sqrt{x}$ $\qquad \dfrac{f(x)}{g(x)} \to \dfrac{0}{0}$, as $x\to0^+$

$g(x)=\sqrt{x}+\sin\sqrt{x}$

$$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{2}\frac{1}{\sqrt{x}}}{\frac{1}{2}\cdot\frac{1}{\sqrt{x}}+\cos\sqrt{x}\cdot\frac{1}{2}\frac{1}{\sqrt{x}}} = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}\cdot\cos\sqrt{x}} = \frac{1}{1+\cos\sqrt{x}} \to \frac{1}{2}, \text{ as } x\to0^+$$

$$\therefore \lim_{x\to0^+}\frac{\sqrt{x}}{\sqrt{x}+\sin\sqrt{x}} = \underline{\frac{1}{2}}$$

18.

$$\lim_{x\to0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x} = \underline{\qquad}$$

$f(x)=\sqrt{a+x}-\sqrt{a-x}$ $\qquad \dfrac{f(x)}{g(x)} \to \dfrac{0}{0}$, as $x\to0$.

$g(x)=x$

$$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{2}\cdot\frac{1}{\sqrt{a+x}} - \frac{1}{2}\cdot\frac{-1}{\sqrt{a-x}}}{1} = \frac{1}{2}\left(\frac{1}{\sqrt{a+x}}+\frac{1}{\sqrt{a-x}}\right) \to \frac{1}{2}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{a}}\right) = \frac{1}{\sqrt{a}}, \text{ as } x\to0$$

$$\therefore \lim_{x\to0}\frac{\sqrt{a+x}-\sqrt{a-x}}{x} = \underline{\frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}}$$

§11-5 (8.

※ 46.

Find a, b for which $\lim_{x\to 0} \dfrac{\sin 2x + ax + bx^3}{x^3} = 0$

<sol>

$$\lim_{x\to 0} \frac{\sin 2x + ax + bx^3}{x^3} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{2\cos 2x + a + 3bx^2}{3x^2} = 0 \quad \because \lim_{x\to 0} 3x^2 = 0 \quad \overset{\text{need}}{\checkmark} \lim_{x\to 0}(2\cos 2x + a + 3bx^2) = 0$$

$$\Rightarrow 2 + a = 0$$

$\Rightarrow a = -2$, the limit exists, so $\lim_{x\to 0} \dfrac{2\cos 2x + 3bx^2 - 2}{3x^2} = 0$

$$\lim_{x\to 0} \frac{2\cos 2x + 3bx^2 - 2}{3x^2} \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{-4\sin 2x + 6bx}{6x} \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{-8\cos 2x + 6b}{6} = 0$$

So, $\lim_{x\to 0}(-8\cos 2x + 6b) = 0 \Rightarrow -8 + 6b = 0 \Rightarrow b = \dfrac{4}{3}$ $\quad \therefore \boxed{a = -2,\ b = \frac{4}{3}}$

※ 49 f: continuous, $\lim_{x\to 0}\left(\dfrac{1}{x} \cdot \int_0^x f(t)\,dt\right) =$ ____

<sol>

$$\lim_{x\to 0} \frac{\int_0^x f(t)\,dt}{x} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0} \frac{\frac{d}{dx}\left(\int_0^x f(t)\,dt\right)}{1} \overset{(\because \text{F.T.C})}{=} \lim_{x\to 0} \frac{f(x)}{1} = \lim_{x\to 0} f(x)$$

$$(\because f\text{: continuous}) = f\left(\lim_{x\to 0} x\right)$$

$$= f(0)$$

$$\therefore \lim_{x\to 0} \frac{\int_0^x f(t)\,dt}{x} = f(0)$$