Hw3 §11-6 ※ 4, 10, 16, 26, 38, 59

§11-7 ※ 2, 16, 22, 28, 34, 38, 56

§11-6

※4.

$$\lim_{x\to\infty} \frac{x^3-1}{2-x} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty} \frac{3x^2}{-1} = -\infty$$

※10.

$$\lim_{x\to\infty} \left(x\cdot\sin\left(\frac{\pi}{x}\right)\right) = \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} = \left(\frac{0}{0}\right) = \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\cdot\frac{-\pi}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \pi\cos\left(\frac{\pi}{x}\right)$$

$$= \pi$$

or

Let $t=\frac{\pi}{x}$ $\quad x\to\infty \Rightarrow t\to 0^+$

$$\lim_{x\to\infty} \left(x\cdot\sin\left(\frac{\pi}{x}\right)\right) = \lim_{x\to\infty} \frac{\pi\cdot\sin\left(\frac{\pi}{x}\right)}{\frac{\pi}{x}} = \lim_{t\to 0^+} \frac{\pi\cdot\sin t}{t} = \pi \quad \left(\because \lim_{t\to 0^+} \frac{\sin t}{t} = 1\right)$$

※16.

$$\lim_{x\to 0} |\sin x|^x = 1$$

$$\ln|\sin x|^x = x\cdot\ln|\sin x| = \frac{\ln|\sin x|}{\frac{1}{x}} \qquad \left(\because \lim_{x\to 0} \frac{x}{x} = 1\right)$$

$$\lim_{x\to 0} \frac{\ln|\sin x|}{\frac{1}{x}} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to 0} \frac{\frac{\cos x}{\sin x}}{\frac{-1}{x^2}} = \lim_{x\to 0} -\frac{x}{\sin x}\cdot x\cdot\cos x = -1\cdot 0\cdot 1 = 0$$

$$\Rightarrow \lim_{x\to 0} \ln|\sin x|^x = \lim_{x\to 0} x\cdot\ln|\sin x| = \lim_{x\to 0} \frac{\ln|\sin x|}{\frac{1}{x}} = 0 \Rightarrow \ln\left(\lim_{x\to 0} |\sin x|^x\right) = 0 \Rightarrow \lim_{x\to 0} |\sin x|^x = e^0$$

$$= 1$$

36.

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}= \underline{e^{ab}}$$

$$\ln\left(1+\frac{a}{x}\right)^{bx}=bx\cdot\ln\left(1+\frac{a}{x}\right)=\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{bx}}$$

$$\lim_{x\to\infty}\ln\left(1+\frac{a}{x}\right)^{bx}=\lim_{x\to\infty}\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{1}{bx}}=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{a}{x}}\cdot\frac{-a}{x^2}}{\frac{-1}{bx^2}}=\lim_{x\to\infty}\frac{1}{1+\frac{a}{x}}\cdot ab=ab$$

$$\Rightarrow \ln\left(\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}\right)=ab \quad\Rightarrow \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=\underline{e^{ab}}$$

38.

$$\lim_{n\to\infty}\frac{\ln n}{n^p}=\left(\frac{\infty}{\infty}\right)=\lim_{n\to\infty}\frac{\frac{1}{n}}{p\cdot n^{p-1}}=\lim_{n\to\infty}\frac{1}{p}\cdot\frac{1}{n^p}=\underline{0}$$

$p>0$

59. $a>0$, $b>0$, show that $\lim_{x\to\infty}\left[\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right]^x=\sqrt{ab}$

<pf>

$$\ln\left[\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right]^x=x\cdot\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)=\frac{\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)}{\frac{1}{x}}$$

$$\lim_{x\to\infty}\ln\left[\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right]^x=\lim_{x\to\infty}\frac{\ln\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)}{\frac{1}{x}}=\left(\frac{0}{0}\right)=\lim_{x\to\infty}\frac{\frac{\cancel{2}}{a^{\frac{1}{x}}+b^{\frac{1}{x}}}\cdot\frac{a^{\frac{1}{x}}\ln a+a^{\frac{1}{x}}\ln b}{\cancel{2}}\cdot\cancel{\frac{-1}{x}}}{\cancel{\frac{-1}{x^2}}}$$

$$=\lim_{x\to\infty}\frac{a^{\frac{1}{x}}\ln a+b^{\frac{1}{x}}\ln b}{a^{\frac{1}{x}}+b^{\frac{1}{x}}}=\frac{\ln a+\ln b}{2}=\frac{1}{2}\ln(ab)=\ln\sqrt{ab}$$

$$\Rightarrow \ln\left(\lim_{x\to\infty}\left[\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right]^x\right)=\ln\left(\sqrt{ab}\right)\Rightarrow \lim_{x\to\infty}\left(\frac{a^{\frac{1}{x}}+b^{\frac{1}{x}}}{2}\right)^x=\sqrt{ab}$$

§11-7

※2. $$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} \tan^{-1}(x)\Big|_{x=0}^{x=b} = \lim_{b\to\infty}\left(\tan^{-1}(b) - \underbrace{\tan^{-1}(0)}_{=0}\right)$$

$$= \lim_{b\to\infty}\tan^{-1}(b) = \frac{\pi}{2}$$

※16. $$\int_e^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\left(1-\frac{1}{\ln b}\right) = 1$$

$$\because \int \frac{1}{x(\ln x)^2}\,dx = \int \frac{1}{(\ln x)^2}\,d(\ln x) = \frac{-1}{\ln x} + c$$

$$\therefore \int_e^b \frac{1}{x(\ln x)^2}\,dx = \frac{-1}{\ln b} - \frac{-1}{\ln e} = 1 - \frac{1}{\ln b}$$

※22. $$\int_{-\infty}^0 x e^x\,dx = \lim_{a\to-\infty}\int_a^0 x e^x\,dx = \lim_{a\to-\infty} a\cdot e^a + e^a - 1 = 0+0-1 = -1$$

$$\because \int x e^x\,dx = \int x\,d(e^x) = x e^x - \int e^x\,dx = x e^x - e^x + c$$

$$\therefore \int_a^0 x e^x\,dx = -1 - (a e^a - e^a) = a e^a + e^a - 1$$

$$\because \lim_{a\to-\infty} e^a = 0$$

$$\lim_{a\to-\infty} a e^a = \lim_{a\to-\infty}\frac{a}{e^{-a}} = \left(\frac{\infty}{\infty}\right) = \lim_{a\to-\infty}\frac{1}{-e^{-a}} = \lim_{a\to-\infty} -e^a = 0$$

$$\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx = \int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$$

$$\int_0^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \lim_{b\to\infty}\int_0^b \frac{e^x}{1+e^{2x}}\,dx = \lim_{b\to\infty}\left(\tan^{-1}(e^b)-\frac{\pi}{4}\right) = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4} \quad \text{—— ①}$$

$$\int_0^b \frac{e^x}{1+e^{2x}}\,dx = \int_1^{e^b} \frac{1}{1+u^2}\,du = \tan^{-1}(u)\Big|_{u=1}^{u=e^b} = \tan^{-1}(e^b) - \tan^{-1}(1)$$

$$= \tan^{-1}(e^b) - \frac{\pi}{4}$$

$u=e^x$, $du=e^x dx$

$x=0 \to u=e^0=1$

$x=b \to u=e^b$

$$\int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx = \int_{\infty}^{0} \frac{e^{-s}}{1+e^{-2s}}(-ds) = \int_0^{\infty} \frac{e^s}{1+e^{2s}}\,ds = \frac{\pi}{4} \quad (\because ①)$$

$s=-x$

$ds=-dx$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx = \frac{\pi}{4}+\frac{\pi}{4} = \underline{\underline{\frac{\pi}{2}}}$$

§11-7

#34

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}dx = \lim_{a\to 0}\int_a^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}dx = \lim_{a\to 0}\left(2-2\sqrt{\sin a}\right) = 2$$

$$\because \int \frac{\cos x}{\sqrt{\sin x}}dx = \int \frac{1}{u}\cdot 2u\,du = \int 2\,du = 2u + C = 2\sqrt{\sin x} + C$$

$u = \sqrt{\sin x}$

$\sin x = u^2$

$\cos x\,dx = 2u\,du$

$$\therefore \int_a^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}dx = 2\sqrt{\sin x}\Big|_a^{\frac{\pi}{2}} = 2 - 2\sqrt{\sin a}$$

#38, (a) $\int_0^{\infty} x^r\cdot e^{-x}dx$ convergent, as $r =$ ____

(b) show by induction that $\int_0^{\infty} x^n\cdot e^{-x}dx = n!$, $n = 1, 2, 3, \cdots$

(a)

$$\int_0^{\infty} x^r\cdot e^{-x}dx = \int_0^1 x^r\cdot e^{-x}dx + \int_1^{\infty} x^r\cdot e^{-x}dx$$

① case 1: $r \le -1$

$\therefore \int_0^1 x^r\cdot e^{-x}dx$ diverges

$\because 0 < x < 1 \Rightarrow 1 < e^x < e \Rightarrow \frac{1}{e} < e^{-x} < 1 \Rightarrow \frac{1}{e}\cdot x^r < x^r\cdot e^{-x} < x^r$

and $\int_0^1 x^r dx = \frac{1}{1+r}x^{r+1}\Big|_0^1$ diverges ($\because r+1$), by comparison test $\Rightarrow \int_0^1 x^r\cdot e^{-x}dx$ diverges

Note: $r = -1$ $\int_0^1 x^{-1}dx = \ln x\Big|_0^1$ diverges.

So, $\int_0^{\infty} x^r\cdot e^{-x}dx$ also diverges.

①

case 2: $r > -1$

why?

$$\int_0^\infty x^r e^{-x} dx = \int_0^{k} x^r \cdot e^{-x} dx + \int_{k}^\infty x^r \cdot e^{-x} dx \text{ converges.}$$

why?

we can find $k>0$ such that $x^r < e^{\frac{x}{2}}$, for $x \geq k$.

($\because$ $e^{\frac{x}{2}}$ grows faster than any power of x)

$$\int_k^\infty x^r \cdot e^{-x} dx = \int_k^\infty \underbrace{\frac{x^r}{e^{x/2}}}_{<1} \cdot e^{-\frac{x}{2}} dx < \int_k^\infty e^{-\frac{x}{2}} dx \quad \text{converges.}$$

as $0 < x < k \Rightarrow 1 < e^x < e^k \Rightarrow e^{-k} < e^{-x} < 1 \Rightarrow e^{-k} \cdot x^r < e^{-x} \cdot x^r < x^r$

$\because \int_0^k x^r dx = \frac{1}{1+r} \cdot x^{1+r} \Big|_0^k$ converges ($\because r+1>0$), by comparison test

$\Rightarrow \int_0^k e^{-x} \cdot x^r dx$ converges. , thus, $\int_0^\infty x^r \cdot e^{-x} dx$ converges, as $r > -1$ #

(b) $n=1$ $\quad \int_0^b x e^{-x} dx = (-xe^{-x} - e^{-x}) \Big|_0^b = -be^{-b} - e^{-b} + 1$

$$\int_0^\infty x e^{-x} dx = \lim_{b\to\infty} \int_0^b x e^{-x} dx = \lim_{b\to\infty} (-b e^{-b} - e^{-b} + 1) = 1$$

assume $\int_0^\infty x^n \cdot e^{-x} dx = n!$ holds.

$$\int_0^\infty x^{n+1} e^{-x} dx = \lim_{b\to\infty} \int_0^b x^{n+1} \cdot e^{-x} dx = \lim_{b\to\infty} \left(-e^{-b} \cdot b^{n+1} + (n+1) \cdot \int_0^b e^{-x} \cdot x^n dx\right)$$

$$\because \int_0^b x^{n+1} e^{-x} dx = \int_0^b -x^{n+1} d(e^{-x}) = -x^{n+1} \cdot e^{-x} \Big|_0^b + \int_0^b e^{-x} \cdot (n+1) \cdot x^n dx$$

$$= -b^{n+1} \cdot e^{-b} + \int_0^b e^{-x} \cdot (n+1) \cdot x^n dx$$

$$= \lim_{b\to\infty} (-b^{n+1} \cdot e^{-b}) + (n+1) \cdot \int_0^\infty e^{-x} \cdot x^n dx = (n+1) \cdot n! = (n+1)!$$

$$\because \lim_{b\to\infty} \frac{-b^{n+1}}{e^{b}} = 0$$

§11-7

※ 15b. $\displaystyle\int_e^\infty \frac{1}{\sqrt{x+1}\cdot \ln x}\,dx$: diverges.

$\because$ $\displaystyle\int_e^\infty \frac{1}{(x+1)\cdot\ln(x+1)}\,dx \leq \int_e^\infty \frac{1}{\sqrt{x+1}\cdot \ln x}\,dx$

$\displaystyle\int_e^\infty \frac{1}{(x+1)\ln(x+1)}\,dx = \int_e^\infty \frac{1}{\ln(1+x)}\,d(\ln(x+1)) = \ln(\ln(x+1))\Big|_e^\infty$: diverges.

$\because$ $x > e > 1 \Rightarrow \ln(x+1) > \ln x \Rightarrow (x+1)\cdot\ln(x+1) \geq (x+1)\cdot\ln x \geq \sqrt{x+1}\cdot\ln x$

$(\ln x > 1 > 0)$