HW 4 · §12-2 ※ 2, 4, 6, 10, 14, 18, 33, 34

§12-3 ※ 6, 10, 16, 19, 30, 32, 38

12-2

※ 2.

$$\sum_{k=3}^{\infty} \frac{1}{k^2-k} = \sum_{k=3}^{\infty} \frac{1}{k(k-1)} = \sum_{k=3}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right) = \lim_{N\to\infty} \sum_{k=3}^{N} \left(\frac{1}{k-1} - \frac{1}{k}\right)$$

$$= \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right)$$

$$= \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{N}\right)$$

$$= \frac{1}{2}$$

※

※ 4.

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+3)} = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+3}\right) \times \frac{1}{2} = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{1}{2} \left(\frac{1}{k+1} - \frac{1}{k+3}\right)$$

$$= \lim_{N\to\infty} \frac{1}{2}\left[\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{N-1}} - \cancel{\frac{1}{N+1}}\right) + \left(\cancel{\frac{1}{N}} - \frac{1}{N+2}\right) + \left(\cancel{\frac{1}{N+1}} - \frac{1}{N+3}\right)\right]$$

$$= \lim_{N\to\infty} \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{N+2} - \frac{1}{N+3}\right]$$

$$= \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$$

※

※ 6.

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{5^k} = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{(-1)^k}{5^k} = \lim_{N\to\infty} \Big(1 - \frac{1}{5} + \frac{1}{25} - \underbrace{\frac{1}{125} + \cdots}_{r=\frac{-1}{5}} + \frac{(-1)^N}{5^N}\Big)$$

$$= \lim_{N\to\infty} \frac{1 \times \left[1 - \left(\frac{-1}{5}\right)^{N+1}\right]}{1 + \frac{1}{5}} = \lim_{N\to\infty} \frac{1 - \left(\frac{-1}{5}\right)^{N+1}}{\frac{6}{5}} = \lim_{N\to\infty} \frac{5}{6}\left[1 - \left(\frac{-1}{5}\right)^{N+1}\right]$$

$$= \frac{5}{6} \quad \left(\because \lim_{N\to\infty} \left(\frac{-1}{5}\right)^{N+1} = 0\right)$$

※

※10

$$\sum_{k=0}^{\infty}\frac{3^{k-1}}{4^{3k+1}} = \lim_{N\to\infty}\sum_{k=0}^{N}\frac{3^{k-1}}{4^{3k+1}} = \lim_{N\to\infty}\sum_{k=0}^{N}\frac{3^{-1}}{4}\cdot\left(\frac{3}{4^3}\right)^k$$

$$= \lim_{N\to\infty}\frac{1}{12}\left[1+\frac{3}{4^3}+\left(\frac{3}{4^3}\right)^2+\cdots\cdots+\left(\frac{3}{4^3}\right)^N\right]$$

$r=\frac{3}{4^3}$

$$= \lim_{N\to\infty}\frac{1}{12}\times\frac{1-\left(\frac{3}{4^3}\right)^{N+1}}{1-\frac{3}{4^3}} \qquad \because \lim_{N\to\infty}\left(\frac{3}{4^3}\right)^{N+1}=0$$

$$= \frac{1}{12}\times\frac{1}{1-\frac{3}{64}} = \frac{1}{\cancel{12}_{3}}\times\frac{\cancel{64}^{16}}{61} = \frac{16}{183}$$

※

※14.

$$\sum_{k=0}^{\infty}(-1)^k\cdot x^{2k} = \frac{1}{1+x^2},\ |x|<1$$

use " if $|x|<1$, then $\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}$ "

$$\sum_{k=0}^{\infty}(-1)^k\cdot x^{2k} = \sum_{k=0}^{\infty}(-x^2)^k = \frac{1}{1-(-x^2)},\ \text{if } |-x^2|<1$$

$$= \frac{1}{1+x^2},\ \text{if } |-x^2|<1 \iff |x|^2<1 \iff |x|<1$$

※

※18.

$$\frac{x}{1+4x^2} = x\cdot\frac{1}{1-(-4x^2)} = x\cdot\sum_{k=0}^{\infty}(-4x^2)^k,\ \text{if } |-4x^2|<1$$

$$\frac{x}{1+4x^2} = \sum_{k=0}^{\infty}(-1)^k\cdot 4^k\cdot x^{2k+1},\ \text{if } |-4x^2|<1 \iff 4|x|^2<1$$

$$\iff |x|^2<\frac{1}{4}$$

$$\iff |x|<\frac{1}{2}$$

§12-2

※33.

Show that $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$ diverges, although $\ln\left(\frac{k+1}{k}\right) \to 0$.

<pf>

$$\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right) = \lim_{N\to\infty} \sum_{k=1}^{N} \ln\left(\frac{k+1}{k}\right) = \lim_{N\to\infty} \sum_{k=1}^{N} \left(\ln(k+1) - \ln k\right)$$

$$= \lim_{N\to\infty} \left[\cancel{\ln 2} - \underbrace{\cancel{\ln 1}}_{=0} + \cancel{\ln 3} - \cancel{\ln 2} + \ln 4 - \cancel{\ln 3} + \cdots + \cancel{\ln N} - \cancel{\ln(N-1)} + \ln(N+1) - \cancel{\ln N}\right]$$

$$= \lim_{N\to\infty} \ln(N+1)$$

(diverges)

$$\lim_{k\to\infty} \ln\left(\frac{k+1}{k}\right) = \ln\left(\lim_{k\to\infty} 1 + \frac{1}{k}\right) = \ln 1 = 0$$

※

※34. Show that $\sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^k$ diverges.

<pf>

$$\lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = e \neq 0$$

So, $\sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^k$ diverges. ($\because$ if $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$

, that is, if $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges)

※

§12-3

※ 6. $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$: converges.

$\because \frac{1}{k^2+1} < \frac{1}{k^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2+1} \leq \sum_{k=1}^{\infty} \frac{1}{k^2}$

$\because 2>1$, the p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges and basic comparison thm.

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2+1}$ converges.

※ 10. $\sum_{k=1}^{\infty} \frac{\ln k}{k^3}$: Converges.

if $k \geq 1 \Rightarrow k \geq \ln k \Rightarrow 1 \geq \frac{\ln k}{k} \Rightarrow \frac{1}{k^2} \geq \frac{\ln k}{k^3} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} \geq \sum_{k=1}^{\infty} \frac{\ln k}{k^3}$

$\because$ the p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges and basic comparison thm.

$\Rightarrow \sum_{k=1}^{\infty} \frac{\ln k}{k^3}$ converges.

§12-3

#16.

$$\sum_{k=2}^{\infty} \frac{2}{k(\ln k)^2} : \underline{\text{converges.}}$$

the integral test:

$$\int_2^{\infty} \frac{2}{x(\ln x)^2}dx = 2\cdot\int_2^{\infty} \frac{1}{x(\ln x)^2}dx = 2\cdot\int_{\ln 2}^{\infty} \frac{1}{u^2}\cdot du = 2\cdot\frac{-1}{u}\Big|_{u=\ln 2}^{u=\infty}$$

$$= \frac{2}{\ln 2} < \infty.$$

$u = \ln x$

$du = \frac{1}{x}dx$

$x=2 \to u=\ln 2$

$x=\infty \to u=\infty$

So, $\sum_{k=2}^{\infty} \frac{2}{k(\ln k)^2}$ converges.

#19 $\sum_{k=1}^{\infty} \frac{2k+5}{5k^3+3k^2}$: $\underline{\text{converges.}}$

$$\frac{2k+5}{5k^3+3k^2} = \frac{1}{k^2}\cdot\frac{2k+5}{5k+3}$$ and $5k+3 \geq 2k+5$, as $k \geq 1 \Rightarrow \frac{2k+5}{5k+3} \leq 1$, as $k \geq 1$

$$\Rightarrow \frac{2k+5}{5k^3+3k^2} \leq \frac{1}{k^2}, \text{ as } k\geq 1 \Rightarrow \sum_{k=1}^{\infty} \frac{2k+5}{5k^3+3k^2} \leq \sum_{k=1}^{\infty}\frac{1}{k^2}$$ and $\sum_{k=1}^{\infty}\frac{1}{k^2}$: converges.

by limit comparison thm.

$$\Rightarrow \sum_{k=1}^{\infty} \frac{2k+5}{5k^3+3k^2}$$ converges.

#30.

$\sum_{k=1}^{\infty} k^2 \cdot 2^{-k^3}$: converges.

the integral test: $\int_1^{\infty} x^2 \cdot 2^{-x^3} dx = \int_1^{\infty} \frac{1}{3} \cdot 2^{-u} du = \frac{1}{3} \int_1^{\infty} 2^{-u} du$

$u = x^3$

$du = 3x^2 dx$

$x=\infty \to u=\infty$

$x=1 \to u=1$

$= \frac{1}{3} \cdot \left(-2^{-u} \cdot \frac{1}{\ln 2}\right) \Big|_{u=1}^{u=\infty}$

$= \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{\ln 2} = \frac{1}{6} \cdot \frac{1}{\ln 2} < \infty$

So, $\sum_{k=1}^{\infty} k^2 \cdot 2^{-k^3}$ converges.

#32.

$\sum_{k=0}^{\infty} \frac{2+\cos k}{\sqrt{k+1}}$: diverges.

$-1 \leq \cos k \leq 1 \Rightarrow 1 \leq 2+\cos k \leq 3 \Rightarrow \frac{1}{\sqrt{k+1}} \leq \frac{2+\cos k}{\sqrt{k+1}}$, as $k \geq 0$

$\Rightarrow \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}} \leq \sum_{k=0}^{\infty} \frac{2+\cos k}{\sqrt{k+1}}$

$\because \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}}$ diverges. and basic comparison thm,

$\Rightarrow \sum_{k=0}^{\infty} \frac{2+\cos k}{\sqrt{k+1}}$ diverges.

§12-3

#38 Find the values of p for which $\sum_{k=2}^{\infty} \frac{\ln k}{k^p}$ converges.

<sol>.

case 1: $p \le 1$, $\sum_{k=3}^{\infty} \frac{\ln k}{k^p} > \sum_{k=3}^{\infty} \frac{1}{k^p}$ ($\because$ as $k \ge 3$, $\ln k \ge 1$)

the p-series $\sum_{k=3}^{\infty} \frac{1}{k^p}$ diverges, because $p \le 1$

$\Rightarrow \sum_{k=3}^{\infty} \frac{\ln k}{k^p}$ diverges.

case 2: $p > 1$, then $\frac{p-1}{2} > 0 \Rightarrow k^{\frac{p-1}{2}} \ge \ln k$, as k: very large.

$\Rightarrow \frac{\ln k}{k^p} \le \frac{k^{\frac{p-1}{2}}}{k^p} = \frac{1}{k^{\frac{p+1}{2}}}$ $\because \frac{p+1}{2} > 1$ $\therefore \sum_{k=2}^{\infty} \frac{1}{k^{\frac{p+1}{2}}}$ converges.

by basic comparison thm,

$\Rightarrow \sum_{k=2}^{\infty} \frac{\ln k}{k^p}$ converges,

Hence, $p > 1 \Leftrightarrow \sum_{k=2}^{\infty} \frac{\ln k}{k^p}$ converges. ∎